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DEPARTMENT OF MATHEMATICS

ENGINEERING MATHEMATICS -I

SUBJECT CODE: MA8151

(Regulation 2017)

UNIT – IV

MULTIPLE INTEGRALS

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4.1Introduction

The mathematical modeling of any engineering problem which leads to the formation of differential equation of more than one variable has its solution by the integration in terms of those variables the need of the solution in an integral where many variables are involved motivated the study of integral calculus of several variables.

In this chapter all the basic concepts related to the methods to approach such integrals are discussed.

4.2Double integration in Cartesian co – ordinates

Let $f(x, y)$ be a single valued function and continuous in a region R bounded by a closed curve C. Let the region R be subdivided in any manner into n sub regions $R_1, R_2, R_3, \cdots, R_n$ of areas $A_1, A_2, A_3, \cdots, A_n$.Let (x_i, y_j) be any point in the sub regionR_i. Then consider the sum formed by multiplying the area of each sub – region by the value of the function $f(x, y)$ at any point of the sub – region and adding up the products which we denote

$$\sum_1^n f(x_i, y_j) A_i$$

The limit of this sum (if it exists) as $n \to \infty$ in such a way that each $A_i \to 0$ is defined as the double integral of $f(x, y)$ over the region R. Thus

$$\lim_{n \to \infty} \sum_1^n f(x_i, y_j) A_i = \iint_R f(x, y)\, dA$$

The above integral can be given as

$$\iint_R f(x, y) dydx \quad or \quad \iint_R f(x, y) dxdy$$

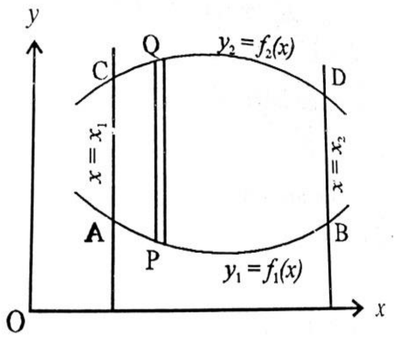
Evaluation of Double Integrals

To evaluate $\int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y) dx\, dy$ we first integrate f(x, y) with respect to x partially, that is treating y as a constant temporarily, between x_0 and x_1. The resulting function got after the inner integration and substitution of limits will be function of y. Then we integrate this function of with respect to y between the limits y_0 and $y_1$as used.

Region of Integration

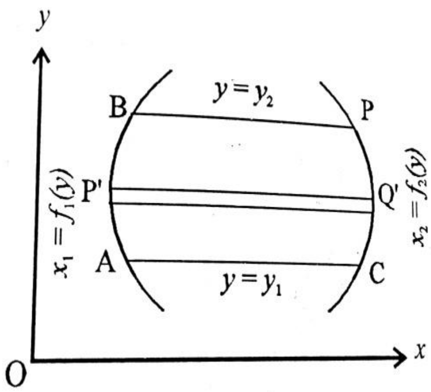
Case (i) Consider the integral $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) dy\, dx$ Given that y varies from $y = f_1(x)$ to $y = f_2(x)$ x varies from $x = a \; to \; x = b$. We get the region R by $y = f_1(x)$,

$y = f_2(x),\ x = a\ ,\ x = b$. The points A, B, C, D are obtained by solving the intersecting curves. Here the region divided into vertical strips $(dy\ dx)$.



Case (ii) Consider the integral $\int_c^d \int_{f_1(y)}^{f_2(y)} f(x,y)dx\ dy$

Here varies from $x = f_1(y)$ to $x = f_2(y)$ and y varies from $y = c\ to\ y = d$ $\therefore$ the region is bounded by $x = f_1(y),\ x = f_2(y),\ y = c\ ,\ y = d$. The points P, Q, R, S are obtained by solving the intersecting curves. Here the region divided into horizontal strips $(dx\ dy)$.



Problems based on Double Integration in Cartesian co-ordinates

Example: 4.1

Evaluate $\int_0^1 \int_1^2 x(x+y)dydx$

Solution:

$$\int_0^1 \int_1^2 x(x+y)dydx = \int_0^1 \int_1^2 (x^2+xy)dydx$$

$$= \int_0^1 \left[x^2 y + \frac{xy^2}{2}\right]_1^2 dx$$

$$= \int_0^1 \left[(2x^2+2x) - (x^2 + \frac{x}{2})\right] dx$$

$$= \int_0^1 \left[2x^2 + 2x - x^2 - \frac{x}{2})\right] dx$$

$$=\int_0^1 \left[x^2 + \frac{3}{2}x\right] dx$$

$$=\left[\frac{x^3}{3} + \frac{3}{2}\frac{x^2}{2}\right]_0^1 \quad =\left(\frac{1}{3} + \frac{3}{4}\right) - (0 + 0) \quad = \frac{13}{12}$$

Example: 4.2

Evaluate $\int_0^a \int_0^b xy(x-y)dydx$

Solution:

$$\int_0^a \int_0^b xy(x-y)dydx \quad = \int_0^a \int_0^b (x^2y - xy^2)dydx$$

$$=\int_0^a \left[\frac{x^2y^2}{2} - \frac{xy^3}{3}\right]_0^b dx$$

$$=\int_0^a \left[\left(\frac{b^2x^2}{2} - \frac{b^3x}{2}\right) - (0-0)\right] dx$$

$$=\left[\left(\frac{b^2x^3}{6} - \frac{b^3x^2}{6}\right)\right]_0^a$$

$$=\left(\frac{a^3b^2}{6} - \frac{a^2b^3}{6}\right) - (0-0)$$

$$=\frac{a^2b^2}{6}(a-b)$$

Example: 4.3

Evaluate $\int_2^a \int_2^b \frac{dxdy}{xy}$

Solution:

$$\int_2^a \int_2^b \frac{dxdy}{xy} \quad = \int_2^a \left[\frac{1}{y} log x\right]_2^b dy$$

$$=\int_2^a \frac{1}{y}(log b - log 2)dy$$

$$= \int_2^a \frac{1}{y} log\left(\frac{b}{2}\right) dy \qquad \left[\because log\frac{a}{b} = log a - log b\right]$$

$$=log\frac{b}{2}\int_2^a \frac{1}{y}dy \quad = \ log\frac{b}{2}[\log y]_2^a$$

$$= \ log\frac{b}{2}[log a - log 2] \quad =\left[log\frac{b}{2}\right]\left[log\frac{a}{2}\right]$$

Example: 4.4

Evaluate $\int_0^1 \int_2^3 (x^2 + y^2)dxdy$

Solution:

$$\int_0^1 \int_2^3 (x^2 + y^2)dxdy \quad = \int_0^1 \left[\frac{x^3}{3} + y^2x\right]_2^3 dy$$

$$=\int_0^1 \left[\left(\frac{3^3}{3} + 3y^2\right) - \left(\frac{2^3}{3} + 2y^2\right)\right] dy$$

$$=\int_0^1 \left[9 + 3y^2 - \frac{8}{3} - 2y^2\right] dy$$

$$=\int_0^1 \left[\frac{19}{3}+y^2\right]dy \quad =\left[\frac{19y}{3}+\frac{y^3}{3}\right]_0^1$$

$$=\left[\frac{19}{3}+\frac{1}{3}\right] \;=\; \frac{20}{3}$$

Example: 4.5

Evaluate $\int_0^3 \int_0^2 e^{x+y}\, dydx$

Solution:

$$\int_0^3 \int_0^2 e^{x+y}\, dydx \;=\int_0^3 \int_0^2 e^x\, e^y dydx \;=\left[\int_0^3 e^x\, dx\right]\left[\int_0^2 e^y\, dy\right]$$

$$=[e^x]_0^3[e^y]_0^2 \quad =[e^3-e^0][e^2-e^0]$$

$$=[e^3-1][e^2-1]$$

Note: If the limits are variable, then check the given problem is in the correct form

Rule: (i) The limits for the inner integral are functions of , then the first integral is with respect to y

(ii) The limits for the inner integral are functions of , then the first integral is with respect to x

Example: 4.6

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} dydx \;=\int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx \quad =\int_0^a\left[\sqrt{a^2-x^2}\right]dx$$

$$=\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right]_0^a$$

$$=\left[\left(0+\frac{a^2}{2}sin^{-1}1\right)-(0+0)\right] \qquad \left[\because sin^{-1}1=\frac{\pi}{2}, sin^{-1}0=0\right]$$

$$=\frac{a^2}{2}\left(\frac{\pi}{2}\right) \;=\; \frac{\pi a^2}{4}$$

Example: 4.7

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y(x^2+y^2)dydx \;=\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y+y^3)dydx$$

$$=\int_0^a \left[\frac{x^2y^2}{2}+\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$=\int_0^a \left[\frac{x^2(a^2-x^2)}{2}+\frac{(a^2-x^2)^2}{4}\right] dx$$

$$=\int_0^a \left[\frac{a^2x^2}{2}-\frac{x^4}{2}+\frac{a^4}{4}+\frac{x^4}{4}-\frac{2a^2x^2}{4}\right] dx$$

$$=\left[\frac{a^2x^3}{6}-\frac{x^5}{10}+\frac{a^4x}{4}+\frac{x^5}{20}-\frac{2a^2x^3}{12}\right]_0^a$$

$$=\left[\frac{-x^5}{10}+\frac{a^4x}{4}+\frac{x^5}{20}\right]_0^a$$

$$=\left[\frac{-a^5}{10}+\frac{a^5}{4}+\frac{a^5}{20}\right]$$

$$=\frac{a^5}{5}$$

Example: 4.8

Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy(x+y)dxdy$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^1 \int_x^{\sqrt{x}} xy(x+y)dydx = \int_0^1 \int_x^{\sqrt{x}} (x^2y+xy^2)dydx$$

$$=\int_0^1 \left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_x^{\sqrt{x}} dx$$

$$=\int_0^1 \left[\left(x^2\frac{x}{2}+x\frac{x^{3/2}}{3}\right)-\left(x^2\frac{x^2}{2}+x\frac{x^3}{3}\right)\right] dx$$

$$=\int_0^1 \left[\frac{x^3}{2}+\frac{x^{5/2}}{3}-\frac{5}{6}x^4\right] dx$$

$$=\left[\frac{x^4}{8}+\frac{x^{7/2}}{3(7/2)}-\frac{5}{6}\frac{x^5}{5}\right]_0^1$$

$$=\left(\frac{1}{8}+\frac{2}{21}-\frac{1}{6}\right)-(0+0-0) = \frac{3}{56}$$

Example: 4.9

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$

Solution:

The given integral is in incorrect form

Thus the correct form is

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{\left(\sqrt{1+x^2}\right)^2+y^2}$$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$=\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}(1) - 0\right] dx \qquad \left[\because tan^{-1}(1) = \frac{\pi}{4}\right]$$

$$=\int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} dx \quad = \frac{\pi}{4}\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx \qquad [tan^{-1}(0) = 0]$$

$$= \frac{\pi}{4}\left[log\left[x+\sqrt{1+x^2}\right]\right]_0^1$$

$$=\frac{\pi}{4}\log(1+\sqrt{2})$$

Example: 4.10

Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dydx$

Solution:

The given integral is in correct form

$$\int_0^4 \int_0^{x^2} e^{y/x}\, dydx = \int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

$$=\int_0^4 \left[\left(\frac{e^x}{1/x}\right) - \left(\frac{1}{1/x}\right)\right] dx$$

$$=\int_0^4 [xe^x - x]dx \;=\int_0^4 x(e^x - 1)\, dx$$

$$=\left[x(e^x - x) - (1)\left(e^x - \frac{x^2}{2}\right)\right]_0^4 \quad \text{(by Bernoulli's formula)}$$

$$=\left[4(e^4 - 4) - \left(e^4 - \frac{16}{2}\right) - (0-1)\right]$$

$$=4e^4 - 16 - e^4 + 8 + 1$$

$$=3e^4 - 7$$

Example: 4.11

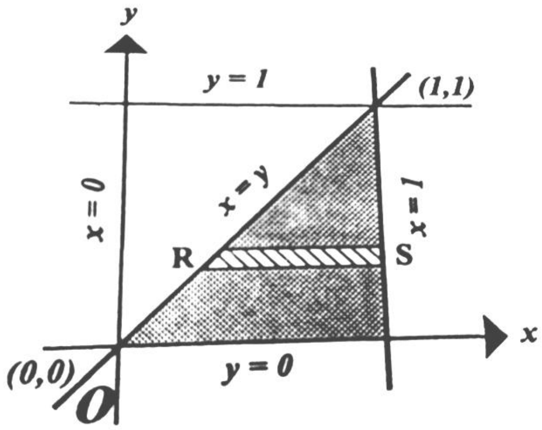
Sketch roughly the region of integration for $\int_0^1 \int_0^x f(x,y)\, dydx$

Solution:

Given $\int_0^1 \int_0^x f(x,y)\, dydx$

x varies from $x = 0$ to $x = 1$

y varies from $y = 0$ to $y = x$



Example: 4.12

Shade the region of integration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy$

Solution:

$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dydx$ is the correct form

x limit varies from $x = 0$ to $x = a$

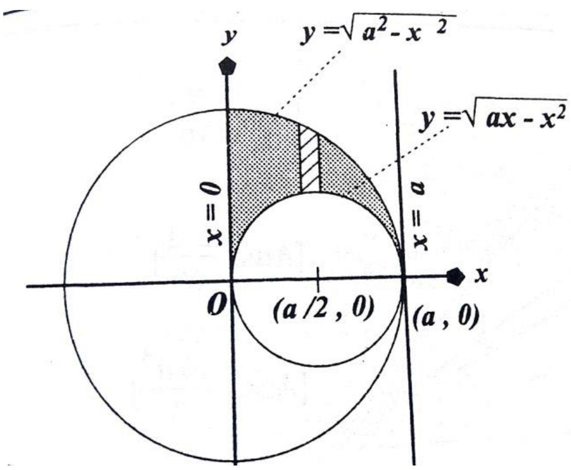
y limit varies from $y = \sqrt{ax - x^2}$ to $y = \sqrt{a^2 - x^2}$

i.e., $y^2 = ax - x^2$ to $y^2 = a^2 - x^2$

i.e., $y^2 + x^2 = ax$ to $y^2 + x^2 = a^2$

$x^2 + y^2 = ax$ is a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$

$x^2 + y^2 = a^2$ is a circle with centre $(0,0)$ and radius a



Exercise 4.1

Evaluate the following integrals

1.$\int_0^1 \int_0^{x^2} (x^2 + y^2)\, dydx$ **Ans:** $\frac{26}{105}$

2. $\int_0^1 \int_x^1 (x^2 + y^2)\, dxdy$ **Ans:** $\frac{1}{3}$

3. $\int_0^a \int_y^a \frac{x}{x^2+y^2} dxdy$ **Ans:** $\frac{\pi a}{4}$

4. $\int_1^2 \int_1^3 (xy^2)\, dxdy$ **Ans:** 13

5. $\int_0^3 \int_1^{\sqrt{4-y}} (x + y)\, dxdy$ **Ans:** $\frac{241}{60}$

6. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$ **Ans:** $1 - \frac{1}{\sqrt{2}}$

7. $\int_0^1 \int_0^x e^{x+y}\, dydx$ **Ans:** $\frac{1}{2}(e - 1)^2$

8. $\int_{-1}^3 \int_{x^2}^{3x+3} dydx$ **Ans:** $\frac{32}{3}$

9. $\int_{-1}^2 \int_{x^2}^{x+2} dydx$ **Ans:** $\frac{9}{2}$

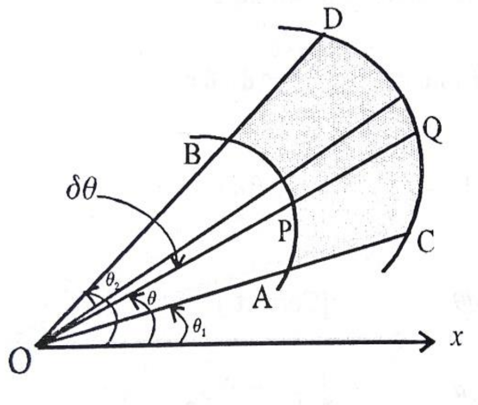
10. $\int_0^{a/\sqrt{2}} \int_0^y (y^2)\, dydx$ **Ans:** $\frac{a^4}{32}(\pi + 2)$

4.3 Double integration in Polar co-ordinates

Consider the integral

$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta) dr\, d\theta$$

which is in polar form. This integral is bounded over the region by the straight line $\theta = \theta_1, \theta = \theta_2$ and the curves $r = r_1, r = r_2$.To evaluate the integral, we first integrate with respect to r between the limits $r = r_1$ and $r = r_2$(treating θ as a constant).The resulting expression is then integrated with respect to θ between the limits $\theta = \theta_1$ and $\theta = \theta_2$.



Geometrically, AB and CD are the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ bounded by the lines $\theta = \theta_1$ and $\theta = \theta_2$ so that ABCD is the region of integration. PQ is a wedge of angular thickness $\delta\theta$.

Then $\int_{r_1}^{r_2} f(r,\theta)dr$ indicates that the integration is performed along PQ(i.e., r varies , θ constant) and the integration with respect to θ

$$\int_{\theta_1}^{\theta_2} f(r,\theta)d\theta$$

means rotation of the strip PQ from AC to BD

Problems based on double integration in Polar Co-ordinates

Example: 4.13

Evaluate $\boldsymbol{\int_0^{\pi/2}\int_0^{sin\theta} rd\theta dr}$

Solution:

Given $\int_0^{\pi/2}\int_0^{sin\theta} rd\theta dr$

$= \int_0^{\pi/2}\int_0^{sin\theta} rdrd\theta$ (Correct form)

$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{sin\theta} d\theta = \int_0^{\pi/2}\left[\frac{(sin\theta)^2}{2} - 0\right]d\theta$

$= \frac{1}{2}\int_0^{\pi/2} sin^2\theta d\theta$

$= \frac{1}{2}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{8}$

Example: 4.14

Evaluate $\boldsymbol{\int_0^{\pi}\int_0^{sin\theta} rdrd\theta}$

Solution:

Given $\boldsymbol{\int_0^{\pi}\int_0^{sin\theta} rdrd\theta}$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{sin\theta} d\theta$

$= \int_0^{\pi}\frac{sin^2\theta}{2}d\theta$

$= \frac{1}{2}\int_0^{\pi}\left[\frac{1-cos2\theta}{2}\right]d\theta$

$= \frac{1}{4}\left[\theta - \frac{sin2\theta}{2}\right]_0^{\pi}$

$= \frac{1}{4}[(\pi - 0) - (0 - 0)]$

$= \frac{\pi}{4}$

Example: 4.15

Evaluate $\boldsymbol{\int_0^{\pi}\int_0^{a} rdrd\theta}$

Solution:

Given $\int_0^{\pi}\int_0^{a} rdrd\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a} d\theta$

$$= \int_0^{\pi} \frac{a^2}{2} d\theta$$

$$= \frac{a^2}{2} [\theta]_0^{\pi}$$

$$= \frac{\pi a^2}{2}$$

Example: 4.16

Evaluate $\int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} r^2 d\theta dr$

Solution:

Given $\int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} r^2 d\theta dr$

$$=\int_{-\pi/2}^{\pi/2} \int_0^{2cos\theta} r^2 dr d\theta \qquad \text{(correct form)}$$

$$=\int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3}\right]_0^{2cos\theta} d\theta$$

$$=\int_{-\pi/2}^{\pi/2} \left[\frac{(2cos\theta)^3}{3} - 0\right] d\theta$$

$$= \frac{8}{3} \int_{-\pi/2}^{\pi/2} cos^3\theta \, d\theta$$

$$= \frac{8}{3} (2) \int_0^{\pi/2} cos^3\theta \, d\theta$$

$$= \frac{16}{3} \left[\frac{2}{3} . 1\right] \qquad = \frac{32}{9}$$

Example: 4.17

Evaluate $\int_0^{\pi/2} \int_{a(1-cos\theta)}^{a} r^2 d\theta dr$

Solution:

Given $\int_0^{\pi/2} \int_{a(1-cos\theta)}^{a} r^2 d\theta dr$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_{a(1-cos\theta)}^{a} d\theta$$

$$= \int_0^{\pi/2} \left[\frac{a^3}{3} - \frac{a^3(1-cos\theta)^3}{3}\right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} [1 - (1 - cos\theta)^3] \mathrm{d}\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} [1 - (1 - 3cos\theta + 3cos^2\theta - cos^3\theta)] \mathrm{d}\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} [3cos\theta + 3cos^2\theta - cos^3\theta)] d\theta$$

$$= \frac{a^3}{3} \left[(3sin\theta)_0^{\pi/2} - 3 \frac{1}{2} \frac{\pi}{2} + \frac{2}{3}\right]$$

$$= \frac{a^3}{3} \left[3 - 3 \frac{\pi}{2} + \frac{2}{3}\right]$$

$= \frac{a^3}{3}\left[\frac{36-9\pi+8}{12}\right]$

$= \frac{a^3}{36}[44 - 9\pi]$

Exercise 4.3

Evaluate the following integrals

1. $\int_0^{\pi/2}\int_{acos\theta}^{a} r^4 dr d\theta$ **Ans:** $\left(\pi - \frac{16}{15}\right)\frac{a^5}{10}$

2. $\int_0^{2\pi}\int_{asin\theta}^{a} r dr d\theta$ **Ans:** $\frac{\pi a^2}{4}$

3. $\int_{-\pi/4}^{\pi/4}\int_0^{cos2\theta} r dr d\theta$ **Ans:** $\frac{\pi}{8}$

4. $\int_0^{\pi/2}\int_0^{acos\theta} r\sqrt{a^2-r^2} dr d\theta$ **Ans:** $\frac{a^3}{18}(3\pi - 4)$

5. $\int_0^{\pi/4}\int_0^{asin\theta} \frac{r}{\sqrt{a^2-r^2}} dr d\theta$ **Ans:** $\frac{a(\pi-3)}{6}$

6. $\int_{b/2}^{b}\int_0^{\pi/2} r d\theta dr$ **Ans:** $\frac{3\pi b^2}{16}$

4.4 Change of order of integration

Change of order of integration is done to make the evaluation of integral easier

The following are very important when the change of order of integration takes place

1. If the limits of the inner integral is a function of x (or function of y) then the first integration should be with respect to y (or with respect to x)
2. Draw the region of integration by using the given limits
3. If the integration is first with respect to x keeping y as a constant then consider the horizontal strip and find the new limits accordingly
4. If the integration is first with respect to y keeping x a constant then consider the vertical strip and find the new limits accordingly
5. After find the new limits evaluate the inner integral first and then the outer integral

Problems

Example:4.18

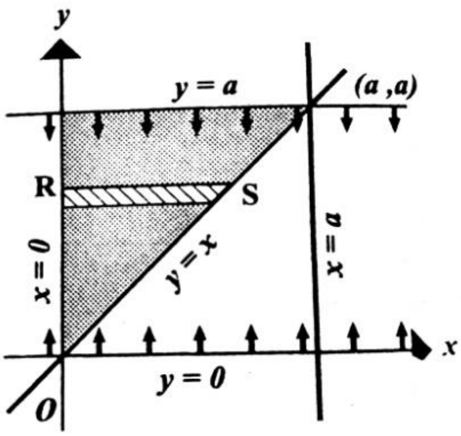
Change the order of integration in $\int_0^a\int_x^a f(x,y)dy\, dx$

Solution:

Given $y: x \to a$

$x: 0 \to a$

The region is bounded by $y = x, y = a, x = 0$ and $x = a$



x axis limit represents the horizontal strip and y axis limit represents vertical

$x: 0 \rightarrow y$

$y: 0 \rightarrow a$

By changing the order we get

$$\int_0^a \int_0^y f(x, y) dx\, dy$$

Example: 4.19

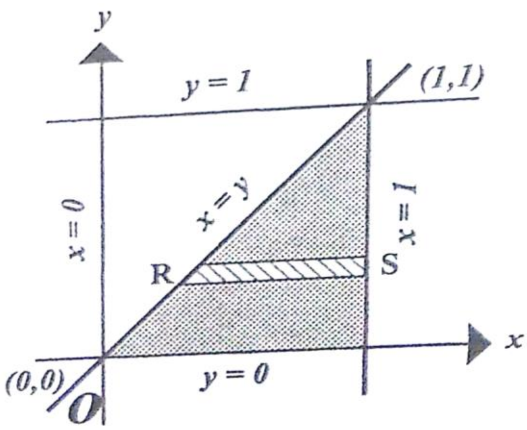
Change the order of integration $\int_0^1 \int_0^x f(x, y) dy\, dx$

Solution:

Given $y: 0 \rightarrow x$

$x: 0 \rightarrow 1$

The region is bounded by $y = 0, y = x, x = 0, x = 1$



$x: y \rightarrow 1$

$y: 0 \rightarrow 1$

By changing the order we get

$$\int_0^1 \int_y^1 f(x,y)dx\,dy$$

Example: 4.20

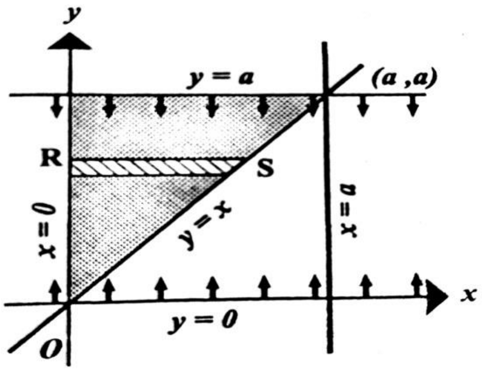
Change the order of integration and hence evaluate it $\int_0^a \int_x^a (x^2+y^2)dy\,dx$

Solution:

It is correct form, given order is $dydx$ given $y: x \to a$

$x: 0 \to a$

the region is bounded by $y = x, y = a\ x = 0$ and $x = a$



x axis limit represent the horizontal strip

y axis limit represents vertical path

changed order is $dx\,dy$

$x: 0 \to y$

$y: 0 \to a$

$$\int_0^a \int_0^y (x^2+y^2)dx\,dy = \int_0^a \left[\frac{x^3}{3} + y^2x\right]_0^y \text{dy}$$

$$= \int_0^a \left[\frac{y^3}{3} + y^3\right] \text{dy}$$

$$= \left[\frac{y^4}{12} + \frac{y^4}{4}\right]_0^a \qquad = \frac{a^4}{12} + \frac{a^4}{4} = \frac{a^4}{3}$$

Example: 4.21

Change the order of integration for $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$

Solution:

It is correct form

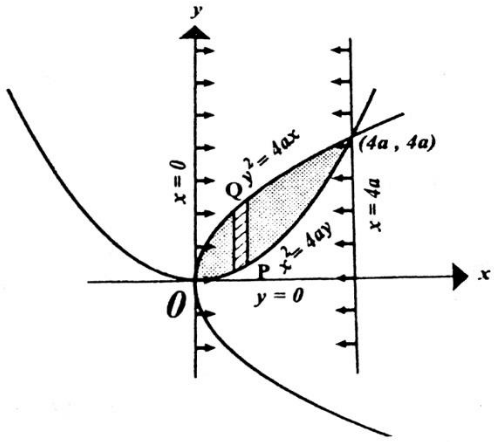
Given order is $dydx$

Given $y: \frac{x^2}{4a} \to 2\sqrt{ax}$

$x: 0 \to 4a$

The region is bounded by $x^2 = 4ay$, $y^2 = 4ax$

$x = 0$ and $x = 4a$



Changed order is $dxdy$ draw a horizontal strip

$x: \frac{y^2}{4a} \to 2\sqrt{ay}$

$y: 0 \to 4a$

$$\int_0^{4a} \int_{y^2/4a}^{2\sqrt{ay}} xy\, dx\, dy = \int_0^{4a} \left[\frac{x^2 y}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left\{\frac{(2\sqrt{ay})^2 y}{2} - \left[\frac{y^2}{4a}\right]^2 \frac{y}{2}\right\} dy$$

$$= \int_0^{4a} \left[\left(\frac{4ay}{2}\right) y - \frac{y^5}{32a^2}\right] dy$$

$$= \left[\frac{4ay^3}{6} - \frac{y^6}{192a^2}\right]_0^{4a}$$

$$= \frac{4a(4a)^3}{6} - \frac{(4a)^6}{192a^2}$$

$$= \frac{128a^4}{3} - \frac{4096}{192} a^4$$

$$= \frac{64a^4}{3}$$

Example: 4.22

Change the order of integration of $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xydxdy$ and hence evaluate it

Solution:

It is correct form

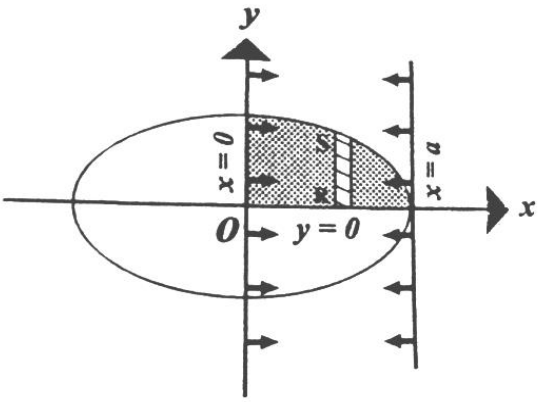
Given order is $dxdy$

Given $x : 0 \to \frac{a}{b}\sqrt{b^2 - y^2}$

$y: 0 \to b$

The region is bounded by $x = 0, x = \frac{a}{b}\sqrt{b^2 - y^2} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y = 0, y = b$



Changed order is $dydx$

Draw the vertical strip

$y : 0 \rightarrow \frac{b}{a}\sqrt{a^2 - x^2}$

$x : 0 \rightarrow a$

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \left[\frac{xy^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{\left[\frac{b}{a}\sqrt{a^2-x^2}\right]^2 x}{2}\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a x(a^2 - x^2)\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a (xa^2 - x^3)\mathrm{dx}$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= b^2\left[\frac{a^4}{2a^2} - \frac{a^4}{4}\right] = b^2\left[\frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^2b^2}{8}$$

Example: 4.23

Change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

Solution:

It is correct form

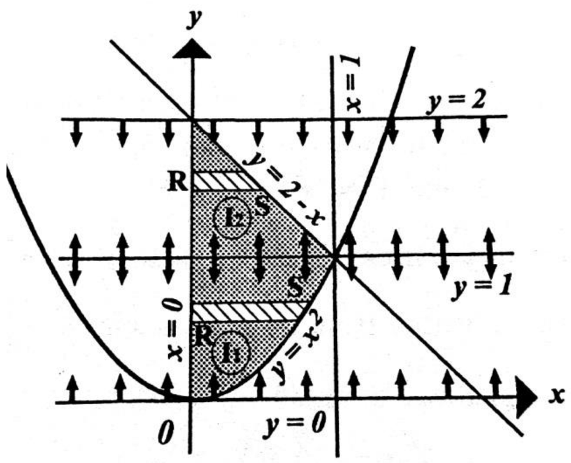
Given order is $dydx$

Given $y: x^2 \to 2-x$

$x: 0 \to 1$

The region is bounded by $= x^2, y + x = 2$

$x = 0, x = 1$



Now divide the region in to two parts i.e. R_1 and R_2

Changed order is $dxdy$

Draw horizontal strip

For Region R_1

Limits are $x: 0 \to \sqrt{y}$

$y: 0 \to 1$

$$\int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy = \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{(\sqrt{y})^2 y}{2} dy$$

$$= \int_0^1 \frac{y^2}{2} \text{dy}$$

$$= \left[\frac{y^3}{6}\right]_0^1$$

$= 1/6$

For region R_2

Limits are $x: 0 \to 2-y$

$y: 1 \to 2$

$$\int_1^2 \int_0^{2-y} xy\, dx\, dy = \int_1^2 \left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

$$= \int_1^2 \frac{(2-y)^2 y}{2}\, dy$$

$= \int_1^2 \frac{(4-4y+y^2)y}{2}\ dy$

$= \frac{1}{2}\ \left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$

$= \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$

$= \frac{5}{24}$

R $= R_1 + R_2$

$= \frac{1}{6} + \frac{5}{24}$

$= \frac{3}{8}$

Example: 4.24

Change the order of integration in $\int_0^1 \int_y^{2-y} xy\ dx\ dy$ and hence evaluates

Solution:

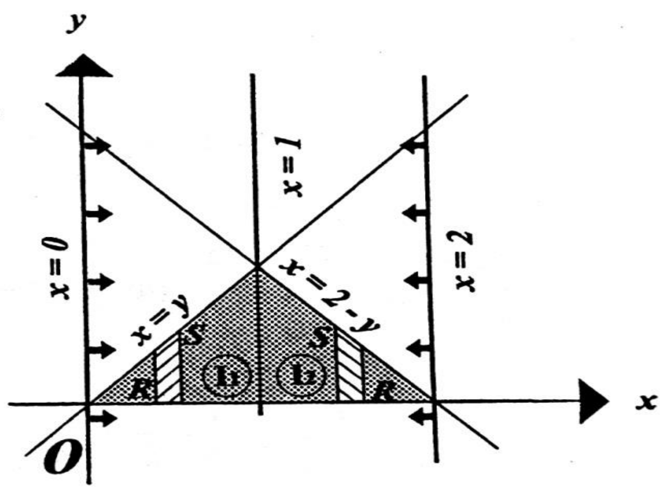
It is correct form

$x:\ y \to 2-y$

$y:\ 0 \to 1$

The region is bounded by $x = y\ , x + y = 2$

$y\ =\ 0\ ,\ \ y\ =\ 1$



Now divide the region into two parts ie. R_1 and R_2

Changed order is $dydx$

Draw horizontal strip

For region R_1

Limits are x: 0$\to$ 1

$y: 0 \rightarrow x$

$$\int_0^1 \int_y^{2-y} xy\,dx\,dy = \int_0^1 \int_0^x xy\,dy\,dx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[\frac{x^3}{3-0}\right] dx$$

$$= \frac{1}{2}\int_0^1 x^3\ dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{8}[x^4]_0^1 = \frac{1}{8}[1-0]$$

$$= \frac{1}{8}$$

For region R_2

$x: 1 \rightarrow 2$

$y: 0 \rightarrow 2-x$

$$\int_1^2 \int_0^{2-x} xy\,dy\,dx = \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \int_1^2 \left[\frac{x\,(2-x)^2}{2} - 0\right] dx$$

$$= \frac{1}{2}\int_1^2 \frac{x\left(4+x^2-4x\right)}{2}\ dx$$

$$= \frac{1}{2}\int_1^2 (4x + x^3 - 4x^2)\,dx$$

$$= \frac{1}{2}\left[4\frac{x^2}{2} + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(8 + \frac{16}{4} - \frac{4}{3}(8)\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}$$

$\Rightarrow R = R_1 + R_2$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$

Example: 4.25

Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ and hence evaluate it

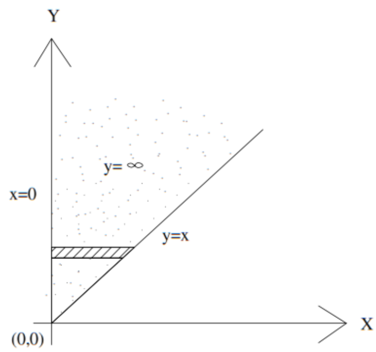
Solution:

It is correct form

Given order is $dy\,dx$

Given $y: \ x \to \infty$

$x: \ 0 \to \ \infty$



Changed order is $dxdy$

Draw a horizontal strip

$x: \ 0 \to y$

$y: \ 0 \to \ \infty$

$\int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\, dy = \ \int_o^\infty \left[e^{-y} \ \frac{x}{y}\right]_0^y dy$

$= \int_o^\infty e^{-y}\, dy \ = \left[\frac{e^{-y}}{1}\right]_0^\infty$

$= -[e^{-\infty} \ - e^0\,] \ = 1$

Example: 4.26

Change the order of integration I = $\int_0^a \int_y^a \frac{x}{x^2+y^2} \boldsymbol{dx\, dy}$ **and the evaluate it**

Solution:

It is correct form

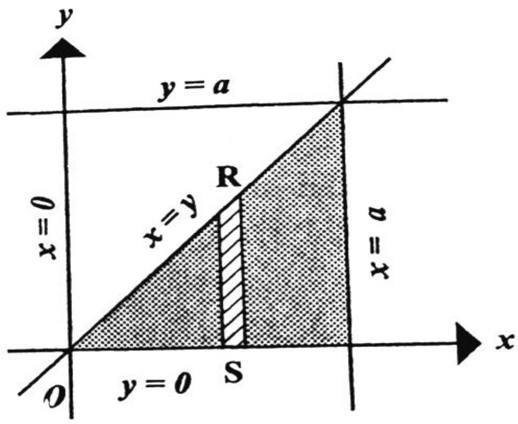
Given order $dxdy$

$x: \ y \to a$

$y: \ 0 \to \ a$

The region is bounded by $x = y\,, x = a$

$y = 0\,, y = a$



Changed order is $dy\, dx$

Draw a vertical strip

$y : 0 \to x$

$x : 0 \to a$

$$\int_0^a \int_0^x \frac{x}{x^2+y^2}\, \text{dy dx} = \int_0^a x \left[\frac{tan^{-1}}{x}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a [tan^{-1}\left(\frac{x}{x}\right) - tan^{-1}0]\, dx$$

$$= \int_0^a \frac{\pi}{4}\, dx$$

$$= \left[\frac{\pi}{4} x\right]_0^a$$

$$= \frac{\pi}{4} a$$

Example: 4.27

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$ by changing the order of integration

Solution:

It is correct form

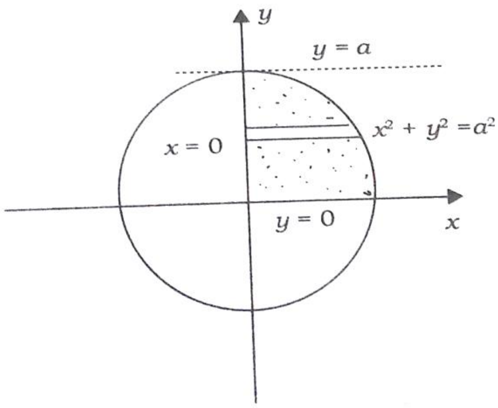
Given order $dy\, dx$

Given $y : 0 \to \sqrt{a^2 - x^2}$

$x : 0 \to a$

the region is bounded by $y = 0,\ y = \sqrt{a^2 - x^2}$

$x = 0\ ,\ x = a$



changed order $dxdy$

Draw horizontal strip

$$x:\ 0 \to \sqrt{a^2 - y^2}$$

$$y:\ 0 \to\ a$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\ dy\ dx = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\ dx\ dy$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a y(a^2\text{-}y^2)\ dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}[\frac{a^2}{2} - \frac{a^4}{4}]$$

$$= \frac{a^4}{8}$$

Exercise: 4.4

Change the order of integration and hence evaluate the following

1. $\int_{-a}^{a} \int_0^{\sqrt{a^2}-x^2} (x^2 + y^2)\ dy\ dx$ **Ans:** $\frac{\pi a^4}{4}$

2. $\int_0^a \int_0^{2\sqrt{ax}} x^2\ \ dy\ dx$ **Ans:** $\frac{4}{7}a^4$

3. $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\ \ dx$ **Ans:** $\frac{16}{3}$

4. $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2}-y^2} dx\, dy$ **Ans:** $\frac{1}{2}$

5. $\int_0^\infty \int_0^y ye^{-\frac{y^2}{x}}$ **Ans:** $\frac{1}{2}$

6. $\int_0^1 \int_y^{2-y} xy\ \ dxdy$ **Ans:** $\frac{1}{3}$

7. $\int_0^1 \int_y^{2-x} \frac{x}{y}\ dydx$ **Ans:** $\log 4 - 1$

8. $\int_1^3 \int_0^{6/x} x^2\ dy\, dx$ **Ans:** 24

9. $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}}\ dx\, dy$ **Ans:** $\frac{a^2}{2}\log(1+\sqrt{2})$

10. $\int_1^4 \int_{2/y}^{2\sqrt{y}} dx\ \ dy$ **Ans:** $\frac{28}{3}$ −2 log 4

4.5 Area enclosed by plane curves (Cartesian coordinates)

Area = $\int\int dy\, dx$ **(or) Area =** $\int\int dxdy$

Example: 4.28

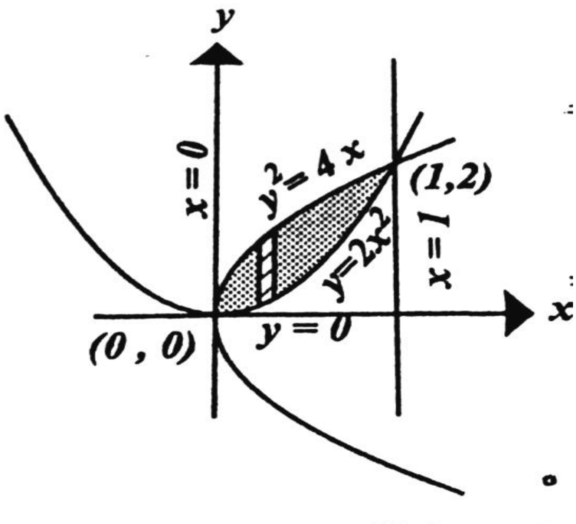
Find the area enclosed by the curves y=2x^2 and $y^2 = 4x$

Solution:

Area = $\int\int dy\, dx$

$y:\ 2x^2 \to 2\sqrt{x}$

$x:\ 0 \to 1$



$$\text{Area} = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy\, dx$$

$$= \int_0^1 [y]_{2x^2}^{2\sqrt{x}} dx$$

$$= \int_0^1 (2\sqrt{x} - 2x^2) dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{2x^3}{3}\right]_0^1$$

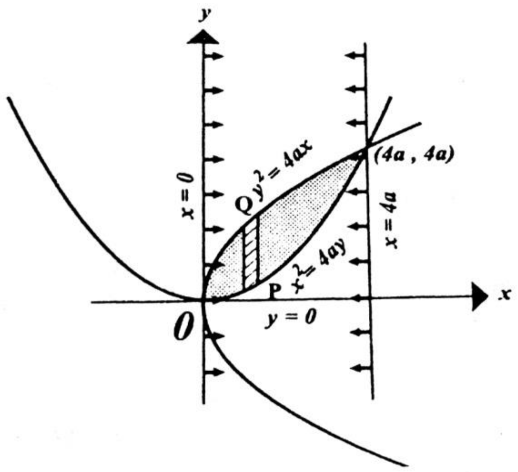
$$= \left[\frac{4x^{3/2}}{3} - \frac{2x^3}{3}\right]_0^1$$

$$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

Example: 4.29

Find the area between the parabola y^2 =4ax and x^2 = 4ay

Solution:



Area $=\int\int dy\,dx$

$$y : \frac{x^2}{4a} \to 2\sqrt{ax}$$

$$x :\ 0 \to\ 4a$$

$$= \int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx$$

$$= \int_0^{4a}[y]_{\frac{x^2}{4a}}^{2\sqrt{ax}}dx$$

$$= \int_0^{4a}(2\sqrt{ax} - \frac{x^2}{4a})\,dx$$

$$= \left[\frac{2\sqrt{a}\ \ x^{3/2}}{3/2} - \frac{x^3}{12a}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{(4a)^3}{12a}$$

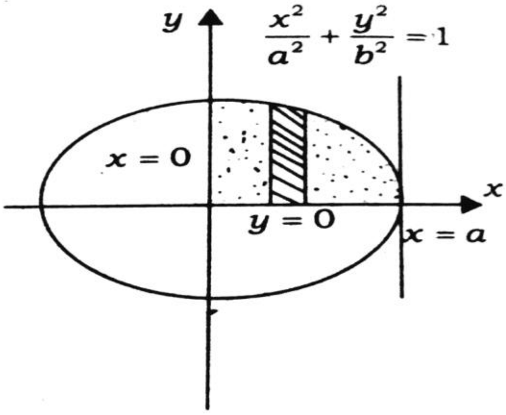
$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Example: 4.30

Find the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ =1

Solution:



Area =$4\iint dx\,dy$

$x: 0 \to \frac{a}{b}\sqrt{b^2-y^2}$

$y: 0 \to ab$

$$\text{Area} = 4\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy\,dx$$

$$= 4\int_0^b [x]_0^{\frac{a}{b}\sqrt{b^2-y^2}} dy$$

$$= 4\int_0^b \left[\frac{a}{b}\sqrt{b^2-y^2} - 0\right] dy$$

$$= \frac{4a}{b}\left[\frac{b^2}{2} sin^{-1}\left(\frac{y}{b}\right) + \frac{y}{2}\sqrt{b^2-y^2}\right]_0^b$$

$$= \frac{4a}{b}[\left(\frac{b^2}{2}\,\frac{\pi}{2} + 0\right) - 0]$$

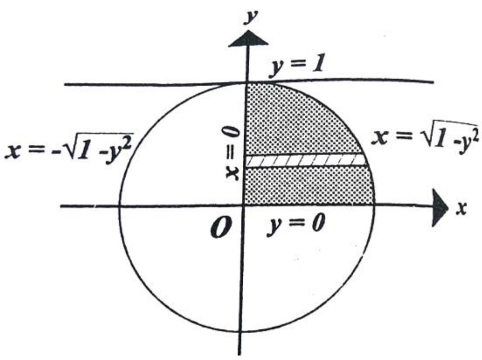
$$= \frac{4ab}{b}\,\frac{b^2}{2}\,\frac{\pi}{2}$$

$$= \pi ab$$

Example: 4.31

Evaluate $\iint xy\;dx\,dy$ over the positive quadrant of the circle $x^2+y^2=1$

Solution:



$x: 0 \to \sqrt{1-y^2}$

$y: 0 \to 1$

$$\iint xy \; dx\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} xy \; dx\,dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{1}{2}\int_0^1 (\sqrt{1-y^2})^2 y \, dy$$

$$= \frac{1}{2}\int_0^1 (1-y^2)\, y \, dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\, dy$$

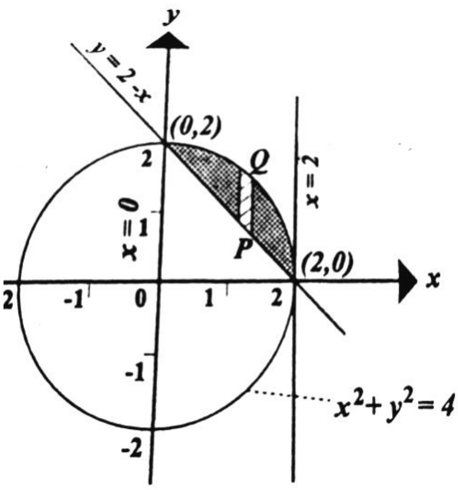
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{2}\left[\frac{1}{4}\right] = \frac{1}{8}$$

Example: 4.32

Find the smaller of the area bounded by $y = 2 - x$ and $x^2 + y^2 = 4$

Solution:



Area= $\iint dy\,dx$

$y: \; 2 - x \to \sqrt{4 - x^2}$

$x: \; 0 \to 2$

$$= \int_0^2 \int_{2-x}^{\sqrt{4-x^2}} dy\,dx$$

$$= \int_0^2 [y]_{2-x}^{\sqrt{4-x^2}} dx$$

$$= \int_0^2 [\sqrt{4-x^2} - (2-x)]dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2x + \frac{x^2}{2}\right]_0^2$$

$$= 0 + \frac{4}{2}\left(\frac{\pi}{2}\right) - 4 + \frac{4^2}{2}$$

$= \pi - 2$ square unit

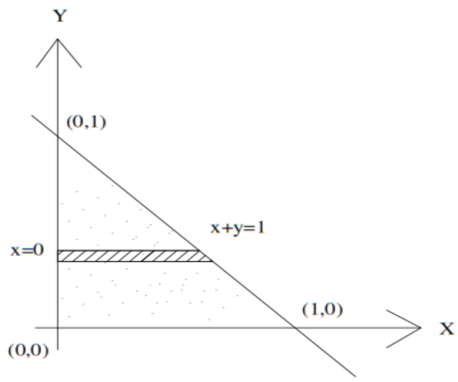
Example: 4.33

Evaluate $\iint$ xy dxdy over the positive quadrant for which $x + y \leq 1$

Solution:

$x:\ 0 \rightarrow 1 - y$

$y:\ 0 \rightarrow\ 1$



$$\iint \text{xy dxdy} = \int_0^1 \int_0^{1-y} \text{xy dx dy}$$

$$= \int_0^1 \left(\frac{x^2 y}{2}\right)_0^{1-y} dy$$

$$= \int_0^1 \frac{1-y^2 y}{2} dy$$

$$= \frac{1}{2}\int_0^1 (y^2 - 2y^2 + y^3) dy$$

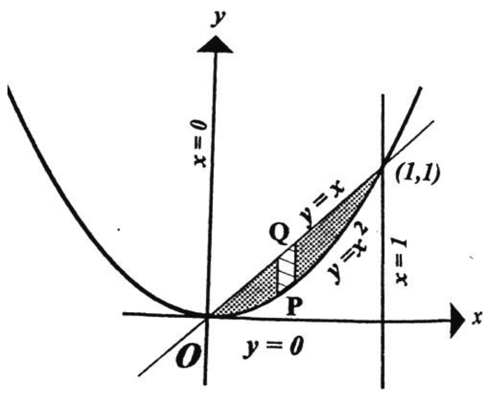
$$= \frac{1}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = \frac{1}{2}\left[\frac{16-8+3}{12}\right] = \frac{1}{24}$$

Example: 4.34

Using double integral find the area bounded by $y = x$ and $y = x^2$

Solution:



Area $= \iint dy\,dx$

$y: x^2 \to x$

$x: 0 \to 1$

$$= \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 [y]_{x^2}^{x}\,dx$$

$$= \int_0^1 (x - x^2)dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

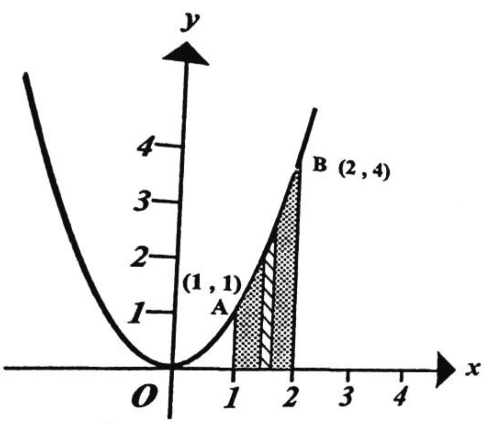
Example: 5.35

Evaluate $\iint(x^2 + y^2)dxdy$ where A is area bounded by the curves x^2=y, x=1 and x=2 about x axis

Solution:

$y: 0 \to x^2$

$x: 1 \to 2$



$$\iint(x^2 + y^2)dxdy = \int_1^2 \int_0^{x^2} (x^2 + y^2)dydx$$

$$= \int_1^2 \left[x^2y + \frac{y^3}{3}\right]_0^{x^2} dx$$

$$= \int_1^2 (x^4 + \frac{x^6}{3})dx$$

$$=\left[\frac{x^5}{5} + \frac{x^7}{21}\right]_1^2$$

$$= [\frac{2^5}{5} + \frac{2^7}{21} - \frac{1}{5} - \frac{1}{21}]$$

$$= \frac{1286}{105}$$

Example: 4.36

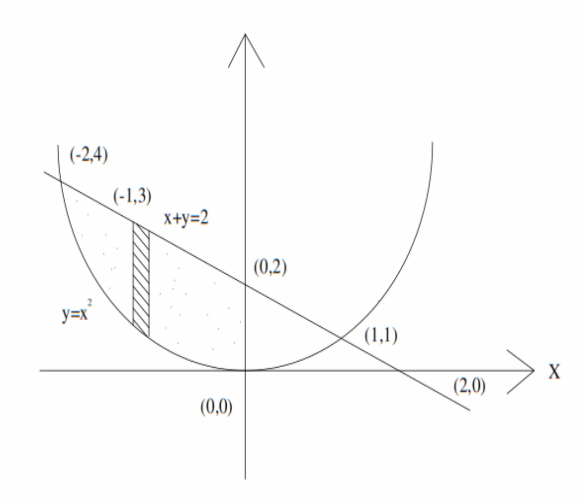
Find the area enclosed by the curves $y = x^2$ and $x + y - 2 = 0$

Solution:

Given $y=x^2$ and $x + y - 2 = 0$

x	0	1	2	-1	-2
Y=$2-x$	2	1	0	3	4

x	0	1	-1	2	-2
$y = x^2$	0	1	1	4	4



Area = $\iint dy\,dx$

$y : x^2 \to 2 - x$

$x : -2 \to 1$

$$\int_{-2}^{1}\int_{x^2}^{2-x} dy\,dx = \int_{-2}^{1}[y]_{x^2}^{2-x}\,dx$$

$$= \int_{-2}^{1}(2 - x - x^2)dx$$

$$= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1}$$

$$= \left[2 - \frac{1}{2} - \frac{1}{3}\right] - \left[-4 - \frac{4}{2} + \frac{8}{3}\right] = \frac{27}{6}$$

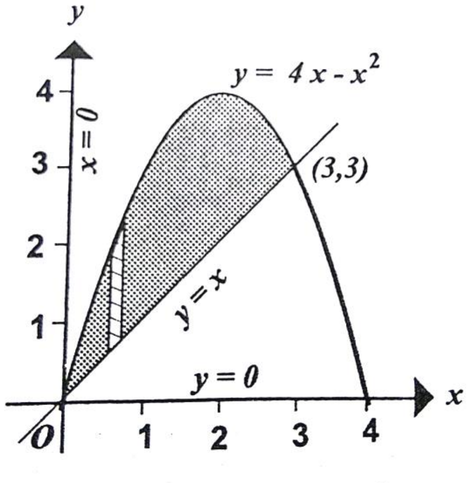
Example: 4.37

Find by double integration the area lying between the parabola y=4x-x^2 and the line y=x

Solution:

Given $y = 4x - x^2$ and $y = x$

x	0	1	2	-1	-2	3
$y = 4x - x^2$	0	3	4	-5	-12	3



Area= $\iint dy\, dx$

$y : x \rightarrow 4x - x^2$

$x : 0 \rightarrow 3$

$$\int_0^3 \int_x^{4x-x^2} dy\, dx = \int_0^3 [y]_x^{4x-x^2} dx$$

$$= \int_0^3 (4x - x^2 - x) dx$$

$$= \int_0^3 (3x - x^2) dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2}$$

Exercise: 4.5

1. Evaluate $\iint$ xdy dx over the region between the parabola $y^2 = x$ and the lines $x + y = 2$, $x = 0$ and $x = 1$ **Ans:** $\frac{4}{15}$

2. Evaluate $\iint y^2\, dxdy$ over the area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **Ans:** $\frac{\pi a b^3}{4}$

3. Find the area between the curve $y^2 = 4 - x$ and the line $y^2 = x$ **Ans:** $\frac{16\sqrt{2}}{3}$

4.6Area Enclosed by Plane Curves [Polar co-ordinates]

Area= $\iint r dr d\theta$

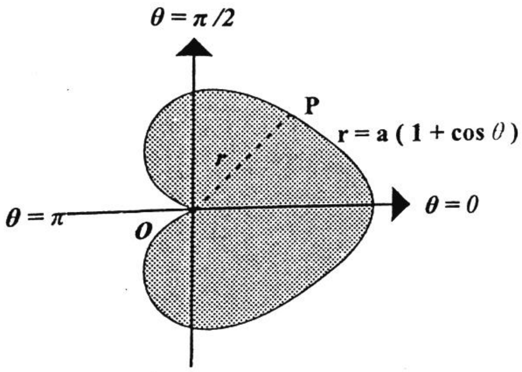
Problems Based on Area Enclosed by Plane Curves Polar Coordinates

Example: 4.38

Find using double integral, the area of the cardioid $r = a(1 + \cos\theta)$

[A.U 2011][A.U 2014][A.U 2015]

Solution:



Area= $\iint \mathbf{rdrd\theta}$

The curve is symmetrical about the initial line

θ varies from $: 0 \to \pi$

r varies from: $0 \to a(1+\cos\theta)$

Hence, required area = $2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a(1+\cos\theta)} rdrd\theta$

$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)} d\theta \quad = \int_0^{\pi}[r^2]_{r=0}^{r=a(1+\cos\theta)} d\theta$

$= \int_0^{\pi}[a^2(1+\cos\theta)^2 - 0]d\theta \;= a^2\int_0^{\pi}[1+\cos^2\theta + 2\cos\theta]d\theta$

$= a^2\int_0^{\pi}\left[1+\frac{1+\cos2\theta}{2}+2\cos\theta\right]d\theta$

$= \frac{a^2}{2}\int_0^{\pi}[2+1+\cos2\theta+4\cos\theta]\,d\theta \;= \frac{a^2}{2}\int_0^{\pi}[3+\cos2\theta+4\cos\theta]\,d\theta$

$= \frac{a^2}{2}\left[3\theta+\frac{\sin2\theta}{2}+4\sin\theta\right]_0^{\pi} \;= \frac{a^2}{2}[(3\pi+0+0)-(0+0+0)]$

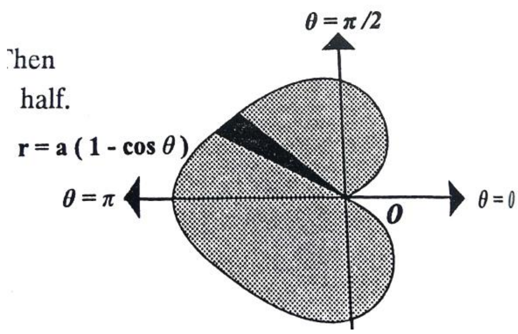
$= \frac{3}{2}a^2\pi$ square units.

Example: 4.39

Find the area of the cardioid $\mathbf{r = a(1-\cos\theta)}$

Solution:

Area= $\iint \mathbf{rdrd\theta}$



The curve is symmetrical about the initial line.

θ varies from : $0 \rightarrow \pi$

r varies from: $0 \rightarrow a(1-\cos\theta)$

Hence, required area $= 2\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a(1-\cos\theta)} rdrd\theta$

$$= 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=a(1-\cos\theta)} d\theta \quad = \int_0^{\pi}[r^2]_{r=0}^{r=a(1-\cos\theta)} d\theta$$

$$= \int_0^{\pi}[a^2(1-\cos\theta)^2 - 0]d\theta \;= a^2\int_0^{\pi}[1+\cos^2\theta - 2\cos\theta]d\theta$$

$$= a^2\int_0^{\pi}\left[1+\frac{1+\cos2\theta}{2} - 2\cos\theta\right]d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}[2+1+\cos2\theta - 4\cos\theta]\,d\theta \;= \frac{a^2}{2}\int_0^{\pi}[3+\cos2\theta-4\cos\theta]\,d\theta$$

$$= \frac{a^2}{2}\left[3\theta + \frac{\sin2\theta}{2} - 4\sin\theta\right]_0^{\pi} \;= \frac{a^2}{2}[(3\pi+0-0)-(0+0-0)]$$

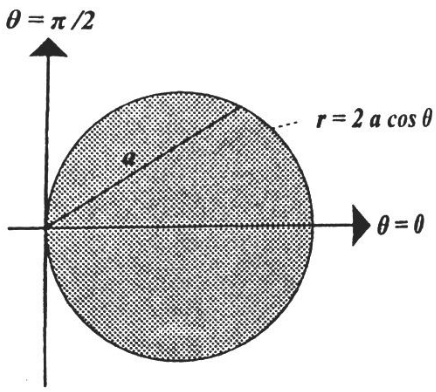
$= \frac{3}{2}a^2\pi$ square units.

Example: 4.40

Find the area of a circle of radius 'a' by double integration in polar co-ordinates.

Solution:

Area= $\iint rdrd\theta$



The equation of circle with pole on the circle and diameter through the point as initial line is $r = 2a\cos\theta$

Area = 2 × upper area

$$= 2\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=2a\cos\theta} rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} [r^2]_0^{2a\cos\theta}\,d\theta$$

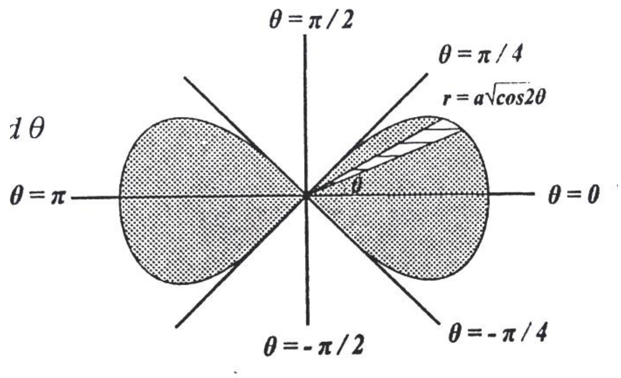
$$= 4a^2\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta \quad = 4a^2.\frac{1}{2}.\frac{\pi}{2} \;= \pi a^2 \text{ square units.}$$

Example: 4.41

Find the area of the lemniscates $r^2 = a^2\cos 2\theta$ by double integration. **[A.U R-08]**

Solution:

Area= $\iint r\,dr\,d\theta$



Area = 4 × area of upper half of one loop.

$$= 4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} (r^2)_0^{a\sqrt{\cos 2\theta}}\,d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= 2a^2\left(\frac{\sin 2\theta}{2}\right)_0^{\frac{\pi}{4}}$$

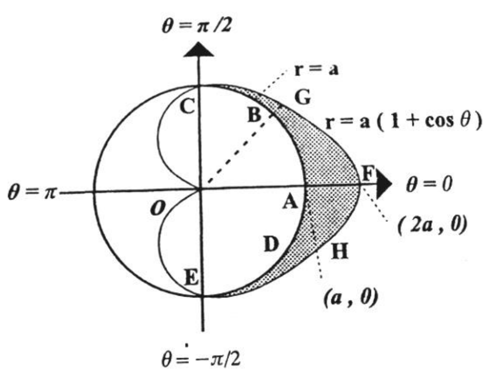
$= a^2$ square units.

Example: 4.42

Find the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$, by double integration. **[A.U 2014]**

Solution:

Area= $\iint r\,dr\,d\theta$



Both the curves are symmetric about the initial line.

Hence, the required area = $2\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=a}^{r=a(1+cos\theta)} rdrd\theta$

$= 2\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{r=a}^{r=a(1+cos\theta)} d\theta$

$= 2\int_0^{\frac{\pi}{2}}\left[\frac{a^2(1+cos\theta)^2}{2}-\frac{a^2}{2}\right] d\theta$

$= a^2\int_0^{\frac{\pi}{2}}[1+cos^2\theta+2cos\,\theta-1]d\theta$

$= a^2\int_0^{\frac{\pi}{2}}\left[\frac{1+cos2\theta}{2}+2cos\theta\right]d\theta$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}[1+cos2\theta+4cos\theta]\,d\theta$

$= \frac{a^2}{2}\left[\theta+\frac{sin2\theta}{2}+4sin\theta\right]_0^{\frac{\pi}{2}}$

$= \frac{a^2}{2}\left[\left(\frac{\pi}{2}+0+4\right)-(0+0+0)\right]$

$= \frac{a^2}{4}(\pi+8)$square units.

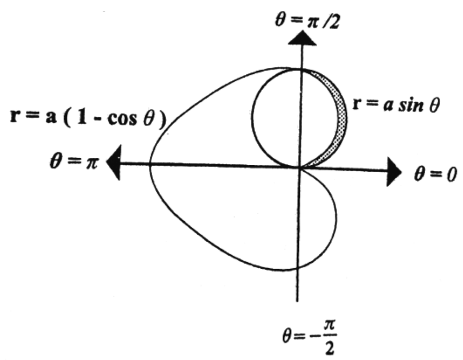
.

Example: 4.43

Find the area inside the circle $\mathbf{r=asin\theta}$ and outside the cardioid $\boldsymbol{r=a(1-\cos\theta)}$

[A.U.Jan.2009]

Solution:



From the figure, we get

θ varies from $: 0\to\frac{\pi}{2}$

r varies from: $a(1-\cos\theta)\to a\sin\theta$

The required area = $\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=a(1-\cos\theta)}^{r=a\sin\theta)} rdrd\theta$

$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{r=a(1-cos\theta)}^{r=asin\theta)} d\theta$

$= \int_0^{\frac{\pi}{2}}\left[\frac{a^2sin^2\theta}{2} - \frac{a^2(1-cos\theta)^2}{2}\right]d\theta$

$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}[sin^2\theta - 1 - cos^2\theta + 2cos\theta]\, d\theta$

$= \frac{a^2}{2}\left[\int_0^{\frac{\pi}{2}} sin^2\theta d\theta - \int_0^{\frac{\pi}{2}} d\theta - \int_0^{\frac{\pi}{2}} cos^2\theta d\theta + 2\int_0^{\frac{\pi}{2}} cos\theta d\theta\right]$

$= \frac{a^2}{2}\left[-[\theta]_0^{\frac{\pi}{2}} + 2[sin\theta]_0^{\frac{\pi}{2}}\right]$ $\qquad \therefore \int_0^{\frac{\pi}{2}} sin^2\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$

$= \frac{a^2}{2}\left[-\left(\frac{\pi}{2} - 0\right) + 2(1-0)\right]$ $\qquad = \frac{1}{2}\frac{\pi}{2}$

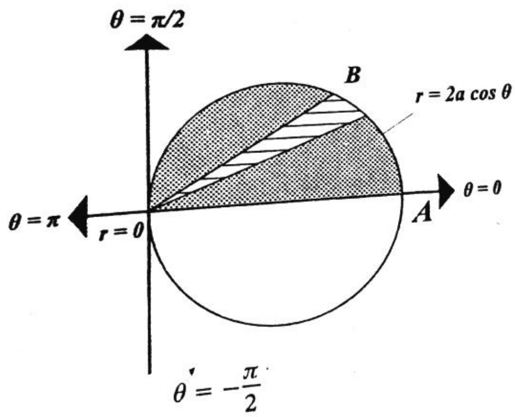
$= \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right]$

$= \frac{a^2}{4}(4-\pi)$ square units.

Example: 4.44

Evaluate $\int_R \int r^2 \sin\theta\, dr d\theta$ where R is the semi circle $r = 2a\cos\theta$ above the initial line.

Solution:



θ varies from $: 0 \to \frac{\pi}{2}$

r varies from: $0 \to 2a\cos\theta$

Let $I = \int\int r^2 sin\theta\, dr d\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta}(r^2 sin\theta) dr d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\, sin\theta\right]_{r=0}^{r=2acos\theta)} d\theta$

$= \int_0^{\frac{\pi}{2}}\left[\frac{8a^3}{3} cos^3\theta\, sin\theta - 0\right] d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{8a^3}{3} cos^3\theta\, sin\theta\, d\theta$

$= \frac{8a^3}{3}\int_0^{\frac{\pi}{2}} cos^3\theta\ sin\theta\ \ d\theta$

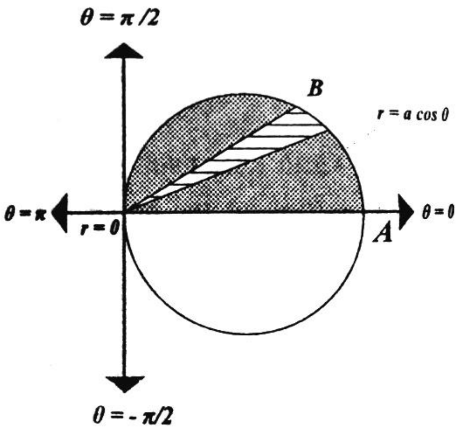
$= \frac{8a^3}{3}\int_0^{\frac{\pi}{2}} cos^3\theta\ \ d(cos\theta)\ \ =\ \ \frac{8a^3}{3}\left[\frac{cos^4\theta}{4}\right]_0^{\frac{\pi}{2}}$

$= \ \frac{8a^3}{3}\left[0-\frac{1}{4}\right]\ \ = \frac{-2a^3}{3}$

Example: 4.45

Evaluate $\int\int r\sqrt{a^2-r^2}\ drd\theta$ over the upper half of the circle $r = acos\theta$.

Solution:



$\theta\ \ varies\ from\ :0 \to \frac{\pi}{2}$

$r\ varies\ from: 0 \to acos\theta$

$Let\ \ \ I = \ \ \int\int r\sqrt{a^2-r^2}\ \ drd\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{acos\theta} r\sqrt{a^2-r^2}\ \ drd\theta$

$Put\ \ a^2 - r^2\ \ =\ \ t^2$ $-2rdr\ \ =\ \ 2tdt$ $-r\ dr\ \ =\ \ t\ dt$ $r\to 0 \quad => t \to a$ $r\to a\ cos\theta \quad => \ t \to asin\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{asin\theta} \sqrt{t^2}\ \ (-tdt)d\theta$

$= -\int_0^{\frac{\pi}{2}}\int_0^{asin\theta} t^2 dtd\theta \quad = \quad = -\int_0^{\frac{\pi}{2}} \left[\frac{t^3}{3}\right]_{r=a}^{r=asin\theta)} \ \ d\theta$

$= -\frac{1}{3}\int_0^{\frac{\pi}{2}}[t^3]_a^{asin\theta}\ d\theta$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} [a^3 sin^3\theta - a^3]\ d\theta$$

$$= -\frac{a^3}{3} \int_0^{\frac{\pi}{2}} [sin^3\theta - 1]\ d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} [1 - sin^3\theta]\ d\theta$$

$$= \frac{a^3}{3} [\theta]_0^{\frac{\pi}{2}} - \frac{a^3}{3} \int_0^{\frac{\pi}{2}} sin^3\theta\ d\theta$$

$$= \frac{a^3}{3} \left[\frac{\pi}{2} - 0\right] - \frac{a^3}{3} \left[\frac{2}{3} . 1\right] \qquad [\because \int_0^{\frac{\pi}{2}} sin^3\theta d\theta = \frac{2}{3}.1\ ]$$

$$= \frac{a^3}{3} \frac{\pi}{2} - \frac{a^3}{3} \frac{2}{3}$$

$$= \frac{a^3}{3} \left[\frac{\pi}{2} - \frac{2}{3}\right]$$

Exercise: 4.6

1. Evaluate $\iint r\sin\theta\ drd\theta$ over the cardioid $r = a(1 - \cos\theta)$ above the initial line.

   Ans: $\frac{4a^2}{3}$

2. Evaluate $\iint \frac{rdrd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscates $r^2 = a^2\cos 2\theta$

   Ans: a $\left(2 - \frac{\pi}{2}\right)$

3. Find by double integration the area bounded by the circles $r = 2\sin\theta$ and $r = 4sin\theta$

   Ans: 3π

4. Find the area outside $r = 2a\cos\theta$ and inside $r = a(1 + \cos\theta)$ **Ans**: $\frac{\pi a^2}{2}$

4.7 Triple Integrals

Triple integration in cartesian co-ordinates is defined over a region R is defined by $\iiint_R f(x, y, z)dxdydz$ or $\iiint_R f(x, y, z)dV$ or $\iiint_R f(x, y, z)d(x, y, z)$.

Type I – Problems on Triple Integrals

Example: 4.46

Evaluate $\int_0^a \int_0^b \int_0^c (x + y + z)dzdydx$

Solution:

$$\int_0^a \int_0^b \int_0^c (x + y + z)dzdydx = \int_0^a \int_0^b \left[xz + yz + \frac{z^2}{2}\right]_0^c dydx$$

$$= \int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2}\right) dydx$$

$$= \int_0^a \left[xcy + \frac{y^2c}{2} + \frac{c^2y}{2}\right]_0^b dx$$

$$= \int_0^a \left(xbc + \frac{b^2c}{2} + \frac{bc^2}{2}\right) dx$$

$$= \left[\frac{x^2bc}{2} + \frac{b^2cx}{2} + \frac{bc^2x}{2}\right]_0^a$$

$$= \left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= \frac{abc}{2}(a + b + c)$$

Example: 4.47

Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy$

Solution:

$$\int_0^1 \int_{y^2}^1 \int_0^{1-x} xdzdxdy = \int_0^1 \int_{y^2}^1 [xz]_0^{1-x}\, dxdy$$

$$= \int_0^1 \int_{y^2}^1 x(1 - x)\, dxdy$$

$$= \int_0^1 \int_{y^2}^1 (x - x^2)\, dxdy$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1 dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3}\right) dy$$

$$= \left[\frac{y}{2} - \frac{y}{3} - \frac{y^5}{10} + \frac{y^7}{21}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{10} + \frac{1}{21}$$

$$= \frac{105-70-21+10}{210} = \frac{24}{210} = \frac{4}{35}$$

Example: 4.48

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dxdydz$

Solution:

$$\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dxdydz = \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z dzdydx$$

$$= \int_0^1 \int_0^{1-x} [e^z]_0^{x+y}\, dydx$$

$$= \int_0^1 \int_0^{1-x} (e^{x+y} - 1)\, dydx$$

$$= \int_0^1 [e^{x+y} - y]_0^{1-x}\, dx$$

$$= \int_0^1 (e^{x+1-x} - 1 + x - e^x)\, dx$$

$$= \int_0^1 (e - 1 + x - e^x)\, dx$$

$$= \left[ex - x + \frac{x^2}{2} - e^x\right]_0^1$$

$$= e - 1 + \frac{1}{2} - e - 0 + 0 - 0 + 1$$
$$= \frac{1}{2}$$

Example: 4.49

Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdxdy$

Solution:

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdxdy = \int_0^a \int_0^{\sqrt{a^2-y^2}} [z]_0^{\sqrt{a^2-x^2-y^2}} dxdy$$
$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\, dxdy$$
$$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{\left(\sqrt{a^2 - y^2}\right)^2 - x^2}\, dxdy$$
$$= \int_0^a \left[\frac{x}{2}\sqrt{(a^2 - y^2) - x^2} + \frac{a^2-y^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$
$$= \int_0^a \left(0 + \frac{a^2-y^2}{2} \sin^{-1}1 - 0 - 0\right) dy$$
$$= \int_0^a \left(\frac{a^2-y^2}{2}\right) \frac{\pi}{2} dy$$
$$= \frac{\pi}{4} \int_0^a (a^2 - y^2)\, dy$$
$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3}\right]_0^a$$
$$= \frac{\pi}{4} \left[a^3 - \frac{a^3}{3} - 0\right]$$
$$= \frac{\pi}{4} \left(\frac{2a^3}{3}\right)$$
$$= \frac{\pi a^3}{6}$$

Example: 4.50

Evaluate $\int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z} dzdydx$

Solution:

$$\int_0^{\log a} \int_0^x \int_0^{x+y} e^{x+y+z} dzdydx = \int_0^{\log a} \int_0^x [e^{x+y+z}]_0^{x+y}\, dydx$$
$$= \int_0^{\log a} \int_0^x (e^{2(x+y)} - e^{x+y})\, dydx$$
$$= \int_0^{\log a} \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^x dx$$
$$= \int_0^{\log a} \left(\frac{e^{4x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x\right) dx$$

$$= \int_0^{\log a}\left(\frac{e^{4x}}{2} - \frac{3}{2}\times\frac{e^{2x}}{2} + e^x\right)dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{8} + e^x\right]_0^{\log a}$$

$$= \frac{e^{4\log a}}{8} - \frac{3}{8}e^{2\log a} + e^{\log a} - \frac{1}{8} + \frac{3}{8} - 1$$

$$= \frac{e^{\log a^4}}{8} - \frac{3}{8}e^{\log a^2} + a + \left(\frac{-1+6-8}{8}\right)$$

$$= \frac{a^4}{8} - \frac{3a^2}{8} + a - \frac{6}{8} \qquad [\because e^{\log x} = x]$$

Type:II Problem on Triple Integral if region is given

Example: 4.51

Express the region $x \geq 0, y \geq 0, z \geq o, x^2 + y^2 + z^2 \leq 1$ by triple integration.

Solution:

For the given region, z varies from 0 to $\sqrt{1 - x^2 - y^2}$

y varies from 0 to $\sqrt{1 - x^2}$

x varies from 0 to 1

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dzdydx$$

Example: 4.52

Evaluate $\iiint x^2yzdxdydz$ taken over the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Solution:

Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Limits are , z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a

$$\iiint x^2yzdxdydz = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2 dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[x^2y\frac{z^2}{2}\right]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left(\frac{x^2yc^2\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a \int_0^{bk} x^2y\left(k - \frac{y}{b}\right)^2 dydx \qquad \left[\because k = 1 - \frac{x}{a}\right]$$

$$= \frac{c^2}{2}\int_0^a\int_0^{bk} x^2 y\left(k^2 + \frac{y^2}{b^2} - \frac{2ky}{b}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a\int_0^{bk} x^2\left(yk^2 + \frac{y^3}{b^2} - \frac{2ky^2}{b}\right) dydx$$

$$= \frac{c^2}{2}\int_0^a x^2\left[\frac{k^2y^2}{2} + \frac{y^4}{4b^2} - \frac{2ky^3}{3b}\right]_0^{bk} dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2} + \frac{b^4k^4}{4b^2} - \frac{2b^3k^4}{3b}\right) dx$$

$$= \frac{c^2}{2}\int_0^a x^2\left(\frac{b^2k^4}{2} + \frac{b^2k^4}{4} - \frac{2b^2k^4}{3}\right) dx$$

$$= \frac{b^2c^2}{2}\int_0^a k^4x^2\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) dx$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1 - \frac{x}{a}\right)^4 dx$$

$$\left[\because (1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\right]$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1 - \frac{4x}{a} + \frac{4\times3}{2!}\times\frac{x^2}{a^2} - \frac{4\times3\times2}{3!}\times\frac{x^3}{a^3} + \frac{4\times3\times2\times\times1}{4!}\times\frac{x^4}{a^4}\right) dx$$

$$= \frac{b^2c^2}{24}\int_0^a\left(x^2 - \frac{4x^3}{a} + \frac{6x^4}{a^2} - \frac{4x^5}{a^3} + \frac{x^6}{a^4}\right) dx$$

$$= \frac{b^2c^2}{24}\left[\frac{x^3}{3} - \frac{4x^4}{4a} + \frac{6x^5}{5a^2} - \frac{4x^6}{6a^3} + \frac{x^7}{7a^4}\right]_0^a$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - \frac{a^4}{a} + \frac{6a^5}{5a^2} - \frac{2a^6}{3a^3} + \frac{a^7}{7a^4}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} - a^3 + \frac{6a^3}{5} - \frac{2a^3}{3} + \frac{a^3}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left[\frac{1}{3} - 1 + \frac{6}{5} - \frac{2}{3} + \frac{1}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{35-105+126-70+15}{105}\right)$$

$$= \frac{a^3b^2c^2}{24}\left(\frac{1}{105}\right)$$

$$= \frac{a^3b^2c^2}{2520}$$

Example: 4.53

Find the value of $\iiint xyzdxdydz$ through the positive spherical octant for which $x^2 + y^2 + z^2 \leq a^2$

Solution:

In the positive octant, the limits are

z varies from 0 to $\sqrt{a^2 - x^2 - y^2}$

y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from 0 to a

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz dz dy dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2 - x^2 - y^2)\, dy dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\, dy dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(\frac{a^2x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4}\right) dx$$

$$= \frac{1}{2}\int_0^a \frac{x}{2}(a^2 - x^2)\left[a^2 - x^2 - \frac{(a^2-x^2)}{2}\right] dx$$

$$= \frac{1}{2}\int_0^a \frac{x(a^2-x^2)(a^2-x^2)}{4} dx$$

$$= \frac{1}{8}\int_0^a x(a^2 - x^2)^2\, dx$$

$Put\ a^2 - x^2 = t$ $\qquad x = 0 \to t = a^2$

$-2xdx = dt$ $\qquad x = a \to t = 0$

$$\Rightarrow \quad I = \frac{1}{8}\int_{a^2}^0 t^2 \left(-\frac{dt}{2}\right)$$

$$= -\frac{1}{16}\int_{a^2}^0 t^2 dt$$

$$= \frac{1}{16}\int_0^{a^2} t^2 dt$$

$$= \frac{1}{16}\left[\frac{t^3}{3}\right]_0^{a^2}$$

$$= \frac{1}{16}\left(\frac{a^6}{3}\right)$$

$$= \frac{a^6}{48}$$

Example: 4.54

Evaluate $\iiint_D (x + y + z)\, dxdydz$ where $D: 1 \le x \le 2, 2 \le y \le 3, 1 \le z \le 3$

Solution:

$$\iiint_D (x + y + z)\, dxdydz = \int_1^2 \int_2^3 \int_1^3 (x + y + z)\, dzdydx$$

$$= \int_1^2 \int_2^3 \left[xz + yz + \frac{z^2}{2}\right]_1^3 dydx$$

$$= \int_1^2 \int_2^3 (3x + 3y + \frac{9}{2} - x - y - \frac{1}{2})\, dydx$$

$$= \int_1^2 \int_2^3 (2x + 2y + 4)\, dydx$$

$$= \int_1^2 \left[2xy + \frac{2y^2}{2} + 4y\right]_2^3 dx$$

$$= \int_1^2 (6x + 9 + 12 - 4x - 4 - 8)\, dx$$

$$= \int_1^2 (2x + 9)\, dx$$

$$= \left[\frac{2x^2}{2} + 9x\right]_1^2$$

$$= 4 + 18 - 1 - 9$$

$$= 12$$

Example: 4.55

Evaluate $\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$ over the first octant of the sphere $x^2 + y^2 + z^2 = a^2$

Solution:

$$\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}} = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dzdydx}{\sqrt{\sqrt{(a^2-x^2-y^2)^2}-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} (\sin^{-1}1 - 0)\, dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\, dydx$$

$$= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}}\, dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}\right]_0^a$$

$$= \frac{\pi}{2} \left[0 + \frac{a^2}{2} \sin^{-1}1 - 0 - 0\right]$$

$$= \frac{\pi}{2} \frac{a^2}{2} \frac{\pi}{2}$$

$$= \frac{\pi^2 a^2}{8}$$

Exercise:4.7

1. Evaluate $\int_0^4 \int_0^1 \int_0^1 (x + y + z)\, dzdydx$ **Ans:**12

2. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} x\, dzdydx$ **Ans:**$\frac{1}{8}$

3. Evaluate $\int_1^3 \int_{\frac{1}{x}}^1 \int_0^{\sqrt{xy}} x\, yzdzdydx$ **Ans:** $\frac{2}{5}\left[\frac{2}{5}\left(9\sqrt{3} - 1\right) - log3\right]$

4. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}dzdydx$ **Ans:** $\frac{5}{8}$

5.Evaluate $\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z}dzdydx$ **Ans:** $\frac{1}{8}[e^{4a}-6e^{2a}+8e^{a}-3]$

6. Evaluate $\iiint_V (x+y+z)\,dxdydz$ where the region V is bounded by

$x+y+z=a\ (a>0), x=0, y=0, z=0$ **Ans:** $\frac{a^4}{8}$

7. Evaluate $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$ **Ans:** $\frac{\pi^2}{8}$

8. Evaluate $\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z}(x+y+z)\,dxdydz$ **Ans:** 0

4.8 Triple Integrals – Volume of Solids

Volume $=\iiint_V dzdydx$ where V is the volume of the given surface.

Example: 4.56

Find the volume of the sphere $x^2+y^2+z^2=a^2$

Solution:

Volume = 8 X volume of the first octant

z varies from 0 to $\sqrt{\mathbf{a^2-x^2-y^2}}$

y varies from 0 to $\sqrt{\mathbf{a^2-x^2}}$

x varies from 0 to a

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}dzdydx$$

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}[Z]_0^{\sqrt{a^2-x^2-y^2}}\,dydx$$

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dydx$$

$$=8\int_0^{a}\int_0^{\sqrt{a^2-x^2}}\sqrt{\left(\sqrt{a^2-x^2}\right)^2-y^2}\,dydx$$

$$=8\int_0^{a}\left[\frac{y}{2}\sqrt{a^2-x^2-y^2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}}dx$$

$$=8\int_0^{a}\left(0+\frac{(a^2-x^2)}{2}\sin^{-1}1-0\right)dx$$

$$=4\int_0^{a}(a^2-x^2)\frac{\pi}{2}dx$$

$$=2\pi\int_0^{a}(a^2-x^2)\,dx$$

$$=2\pi\left[a^2x-\frac{x^3}{3}\right]_0^{a}$$

$$=2\pi\times\frac{2a^3}{3}$$

$$= \frac{4\pi a^3}{3} \text{ cu. units.}$$

Example: 4.57

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:

Volume =8 X volume of the first octant

z varies from 0 to $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

y varies from 0 to $\sqrt{1 - \frac{x^2}{a^2}}$

x varies from 0 to a

$$V = 8\int_0^a \int_0^{\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [Z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{\left(1 - \frac{x^2}{a^2}\right) - \frac{y^2}{b^2}}\, dydx$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dydx$$

$$= \frac{8c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right) - y^2}\, dydx$$

$$= \frac{8c}{b}\int_0^a \int_0^k \sqrt{k^2 - y^2}\, dydx \qquad \textit{where } k^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{k^2 - y^2} + \frac{k^2}{2} sin^{-1}\frac{y}{k}\right]_0^k dx$$

$$= \frac{8c}{b}\int_0^a \left(0 + \frac{k^2}{2} sin^{-1}1 - 0\right) dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{k^2}{2}\right)\frac{\pi}{2} dx$$

$$= \frac{2c\pi}{b}\int_0^a k^2\, dx$$

$$= \frac{2c\pi}{b}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi\int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi\left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2bc\pi\left[a - \frac{a^3}{3a^2}\right]$$

$$= 2bc\pi\left(a - \frac{a}{3}\right)$$

$$= 2bc\pi \times \frac{2a}{3}$$

$$= \frac{4\pi abc}{3}\ cu.units.$$

Example: 4.58

Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution:

Volume = $\iiint_V \ dzdydx$

z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a

$$V = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [Z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dydx$$

$$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a b\left[\left(1 - \frac{x}{a}\right) - \frac{x}{a}\left(1 - \frac{x}{a}\right) - \frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right] dx$$

$$= bc\int_0^a \left[\left(1 - \frac{x}{a}\right)^2 - \frac{1}{2}\left(1 - \frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2} \int_0^a \left(1 - \frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1 - \frac{x}{a}\right)^3\right]_0^a$$

$$= -\frac{abc}{6}(0 - 1)$$

$$= \frac{abc}{6}\ \text{cu. units.}$$

Example: 4.59

Evaluate $\iiint_V \ dxdydz$ where V is the volume enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2 - x$.

Solution:

In the positive octant, the limits are

z varies from 0 to $2-x$

x varies from 0 to $\sqrt{1-y^2}$

y varies from -1 to 1

$$\iiint dxdydz = 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{2-x} dzdxdy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}[z]_{0}^{2-x}\, dxdy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}(2-x)\, dxdy$$

$$= 2\int_{-1}^{1}\left[2x-\frac{x^2}{2}\right]_{0}^{\sqrt{1-y^2}} dy$$

$$= 2\int_{-1}^{1}\left[2\sqrt{1-y^2}-\left(\frac{1-y^2}{2}\right)\right] dy$$

$$= 4\int_{-1}^{1}\left[\sqrt{1-y^2}\right] dy - \int_{-1}^{1}[1-y^2]\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\sin^{-1}y\right]_{-1}^{1} - \left[y-\frac{y^3}{3}\right]_{-1}^{1}$$

$$= 4\left[0+\frac{1}{2}\sin^{-1}1-0-\frac{1}{2}\sin^{-1}(-1)\right]-\left[1-\frac{1}{3}+1-\frac{1}{3}\right]$$

$$= 4\left[\frac{1}{2}\frac{\pi}{2}+\frac{1}{2}\frac{\pi}{2}\right]-\left[2-\frac{2}{3}\right]$$

$$= 4\left(\frac{2\pi}{4}\right)-\frac{4}{3}$$

$$= 2\pi-\frac{4}{3}$$

Exercise: 4.8

1. Find the volume of the tetrahedron whose vertices are (0,0,0), (0,1,0), (1,0,0) and (0,0,1)

Ans: $\frac{1}{6}$ cu. units.

2. Evaluate $\iiint_V dzdxdy$, where V is the volume enclosed by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$. **Ans:** *1* 6πcu. units

3. Find the volume of the region bounded by the paraboloid $z=x^2+y^2$ and the plane $z=4$

Ans: 8πcu. units

4. Find the volume of the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$ by using triple integration.

Ans: 3 2πcu. units

5. Find the volume of the tetrahedron bounded by coordinate planes and the plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ **Ans:** 4cu. units

4.9 Change of variables in Double integrals:

4.9(a) Evaluation of double integrals by changing Cartesian to polar co-ordinates:

Working rule:

Step:1

Check the given order whether it is correct or not.

Step:2

Write the equations by using given limits.

Step:3

By using the equations sketch the region of integration.

Step:4

Replacement: put $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$

Step:5

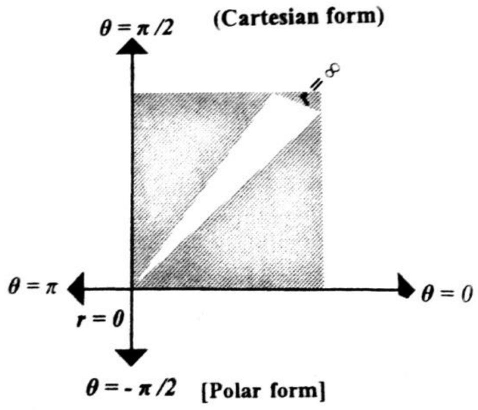
Find r limits(draw radial strip inside the region) and θ limits and evaluate the integral.

Example: 4.60

Change into polar co-ordinates and then evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx$

[AU June 2011,Dec2005]

Solution:



Given order $dydx$ is in correct form.

Given limits are $y : 0 \to \infty , x : 0 \to \infty$

Equations are $y = 0 , y = \infty , x = 0 , x = \infty$

Replacement:

Put $x^2 + y^2 = r^2$, dydx = $rdrd\theta$

Limits:

$r : 0 \to \infty$, $\theta : 0 \to \frac{\pi}{2}$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta$$

Substitution: Put $r^2 = t$, if r = 0 ⇒ $t = 0$, $r = \infty \Rightarrow t = \infty$

$2rdr = dt$ $t: 0 \to \infty$

$rdr = \frac{dt}{2}$

$$\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-t} \frac{dt}{2} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (-e^{-\infty} + e^0) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (0 + 1) d\theta \qquad (\because e^{-\infty} = 0, e^0 = 1)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2} (\theta)_0^{\frac{\pi}{2}}$$

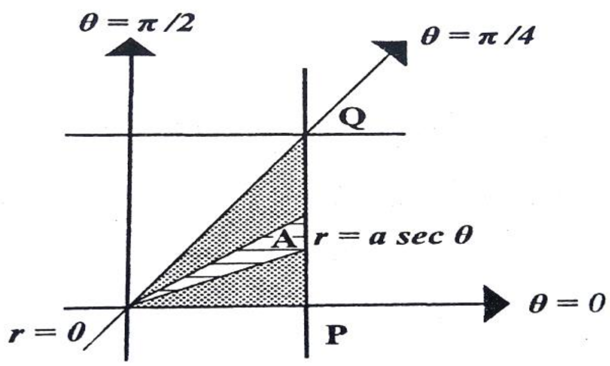
$$= \frac{1}{2} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

Example: 4.61

Change into polar co-ordinates and then evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2} dydx$

Solution:



Given order $dxdy$ is in correct form.

Given limits are $x : y \to a$, $y : 0 \to a$

Equations are $x = y$, $x = a$, $y = 0$, $y = a$

Replacement:

Put $x = r\cos\theta$, $x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

Limits: $r : 0 \to \frac{a}{cos\theta}$, $\theta: 0 \to \frac{\pi}{4}$

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dydx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{cos\theta}} \frac{rcos\theta}{r^2} rdrd\theta$$

$$= \int_0^{\frac{\pi}{4}} [rcos\theta]_0^{\frac{a}{cos\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} (\frac{a}{cos\theta} cos\theta - 0)\, d\theta$$

$$= a \int_0^{\frac{\pi}{4}} d\theta$$

$$= a(\theta)_0^{\frac{\pi}{4}}$$

$$= a(\frac{\pi}{4} - 0)$$

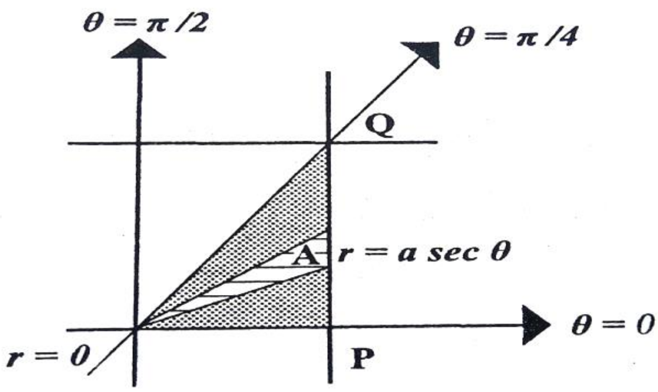
$$= \frac{a\pi}{4}$$

Example: 4.62

Evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}$ dxdy by changing into polar co-ordinates.

[AU Apr 2009, May 2005,Nov 1998]

Solution:



Given order $dxdy$ is in correct form.

Given limits are $x : y \to a$, $y : 0 \to a$

Equations are $x = y$, $x = a$, $y = 0$, $y = a$

Replacement:

Put $x^2 = r^2cos^2\theta$, $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$, $dxdy = rdrd\theta$

Limits: $r : 0 \to \frac{a}{cos\theta}$, $\theta: 0 \to \frac{\pi}{4}$

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dxdy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{cos\theta}} \frac{r^2cos^2\theta}{r} rdrd\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3} cos^2\theta\right]_0^{\frac{a}{cos\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{a^3}{3cos^3\theta} cos^2\theta - 0\right) d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \frac{1}{cos^3\theta} cos^2\theta \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \frac{1}{cos\theta} d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} sec\theta \, d\theta$$

$$= \frac{a^3}{3} (log(sec\theta + tan\theta))_0^{\frac{\pi}{4}}$$

$$= \frac{a^3}{3} \left[log\left(sec\frac{\pi}{4} + tan\frac{\pi}{4}\right) - log(sec0 + tan0)\right]$$

$$= \frac{a^3}{3} \left[log(\sqrt{2} + 1) - log(1 - 0)\right]$$

$$= \frac{a^3}{3} log(\sqrt{2} + 1)$$

Note:

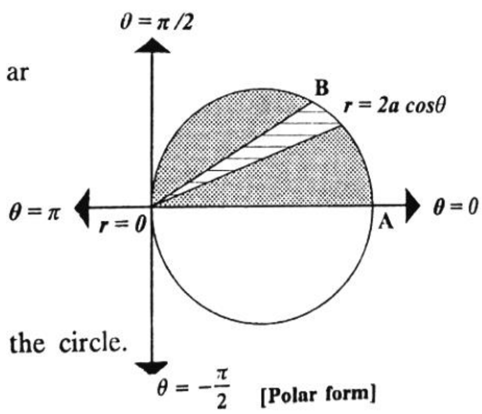
1. $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$
2. $\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta d\theta = \frac{1}{2} \times \frac{\pi}{2}$
3. $\int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \int_0^{\frac{\pi}{2}} \sin^4\theta d\theta = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$
4. $\int_0^{\frac{\pi}{2}} \cos^2\theta\sin^2\theta d\theta = \frac{1}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

Example: 4.63

By changing into polar co-ordinates and evaluate $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2 + y^2) dydx$

[AU Dec 1999, AU A/M 2011]

Solution:



Given order $dydx$ is in correct form.

Given limits are $y : 0 \to \sqrt{2ax - x^2}$, $x : 0 \to 2a$

Equations are $y = 0$, $y = \sqrt{2ax - x^2}$, $x = 0$, $x = 2a$

$$y^2 = 2ax - x^2$$

$x^2 + y^2 - 2ax = 0$ is a circle with centre (a,0) and radius 'a'.

Replacement:

Put $x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

Limits: $r : 0 \to 2acos\theta$, $\theta : 0 \to \frac{\pi}{2}$

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)dydx = \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} r^2 \times rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} r^3 drd\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2acos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{2^4a^4\cos^4\theta}{4} - 0\right) d\theta$$

$$= 4a^4\int_0^{\frac{\pi}{2}}\cos^4\theta\, d\theta$$

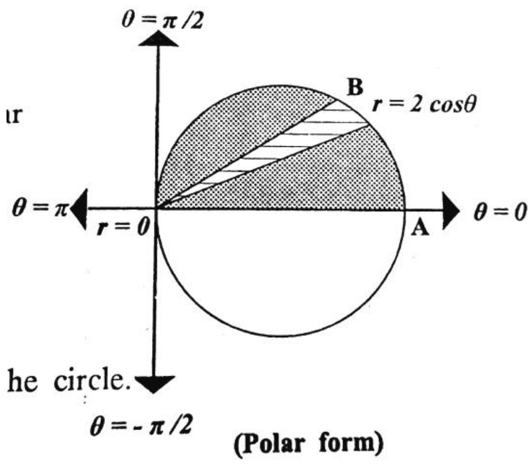
$$= 4a^4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} \qquad \left(\because \int_0^{\frac{\pi}{2}}\cos^4\theta d\theta = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}\right)$$

$$= \frac{3\pi a^4}{4}$$

Example: 4.64

By changing into polar co-ordinates and evaluate $\int_0^2\int_0^{\sqrt{2x-x^2}}\frac{x}{x^2+y^2}dxdy$

Solution:



Given order $dxdy$ is in incorrect form.

The correct form is $dydx \Rightarrow \int_0^2\int_0^{\sqrt{2x-x^2}}\frac{x}{x^2+y^2}dydx$

Given limits are $y : 0 \to \sqrt{2x - x^2}$, $x : 0 \to 2$

Equations are $y = 0$, $y = \sqrt{2x - x^2}$, $x = 0$, $x = 2$

$$y^2 = 2x - x^2$$

$x^2 + y^2 - 2x = 0$ is a circle with centre (1,0) and radius '1'.

Replacement:

Put $x = rcos\theta,\ x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

Limits: $r : 0 \to 2cos\theta$, $\theta : 0 \to \frac{\pi}{2}$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2} dydx = \int_0^{\frac{\pi}{2}} \int_0^{2cos\theta} \frac{rcos\theta}{r^2} \times rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} [rcos\theta]_0^{2cos\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} (2cos^2\theta - 0)\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} cos^2\theta\, d\theta$$

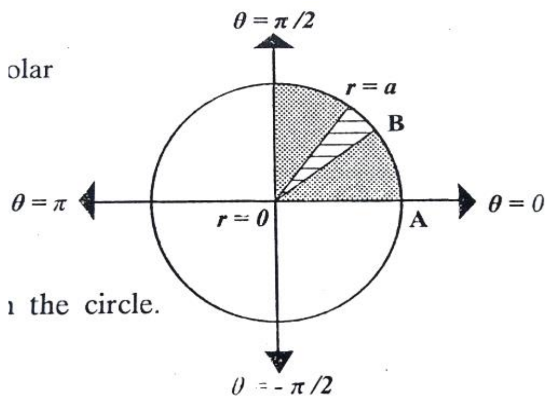
$$= 2 \times \frac{1}{2} \times \frac{\pi}{2} \qquad (\because \int_0^{\frac{\pi}{2}} cos^2\theta d\theta = \frac{1}{2} \times \frac{\pi}{2})$$

$$= \frac{\pi}{2}$$

Example: 4.65

By changing into polar co-ordinates and evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dydx$

Solution:



Given order $dxdy$ is in correct form.

Given limits are $y : 0 \to \sqrt{a^2 - x^2}$, $x : 0 \to a$

Equations are $y = 0$, $y = \sqrt{a^2 - x^2}$, $x = 0$, $x = a$

$$y^2 = a^2 - x^2$$

$x^2 + y^2 = a^2$ is a circle with centre (0,0) and radius 'a'.

Replacement:

Put $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$, $dydx = rdrd\theta$

Limits: $r : 0 \to a$, $\theta: 0 \to \frac{\pi}{2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dydx = \int_0^{\frac{\pi}{2}} \int_0^a r \times r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^2 dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a^3}{3} - 0\right) d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{a^3}{3} (\theta)_0^{\frac{\pi}{2}}$$

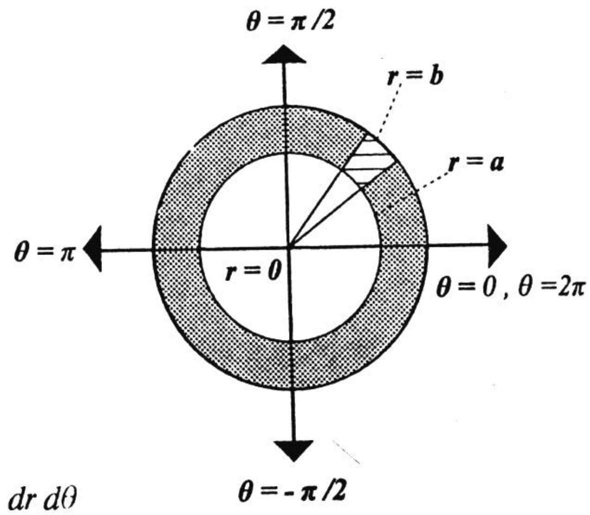
$$= \frac{a^3}{3} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi a^3}{6}$$

Example: 4.66

Evaluate $\iint \frac{x^2y^2}{x^2+y^2} dxdy$ over the annular region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ (b>a) by transforming into polar co-ordinates.

Solution:



Replacement:

Put $x^2 = r^2\cos^2\theta, y^2 = r^2\sin^2\theta$

$x^2 + y^2 = r^2, dxdy = rdrd\theta$

Given the region is between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$

Limits: $r : a \to b$, $\theta: 0 \to 2\pi$

$$\therefore \iint \frac{x^2y^2}{x^2+y^2} dxdy = \int_0^{2\pi} \int_a^b \frac{r^2cos^2\theta \times r^2 sin^2\theta}{r^2} \times rdrd\theta$$

$$= \int_0^{2\pi} \int_a^b \frac{r^5 \cos^2\theta \times \sin^2\theta}{r^2} \times drd\theta$$

$$= \int_0^{2\pi} \int_a^b r^3 \cos^2\theta \times \sin^2\theta \, drd\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_a^b \cos^2\theta \times \sin^2\theta \, d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} (b^4 - a^4) \cos^2\theta \times \sin^2\theta \, d\theta$$

$$= \frac{(b^4 - a^4)}{4} \int_0^{2\pi} \cos^2\theta \times \sin^2\theta \, d\theta$$

$$= \frac{(b^4 - a^4)}{4} 4 \times \int_0^{\frac{\pi}{2}} \cos^2\theta \times \sin^2\theta \, d\theta \quad (\because \int_0^{2\pi} = 4\int_0^{\frac{\pi}{2}})$$

$$= (b^4 - a^4) \times \int_0^{\frac{\pi}{2}} \cos^2\theta \times \sin^2\theta \, d\theta$$

$$= (b^4 - a^4) \times \frac{1}{4} \times \frac{1}{2} \times \frac{\pi}{2} \quad (\because \int_0^{\frac{\pi}{2}} \cos^2\theta \sin^2\theta d\theta = \frac{1}{4} \times \frac{1}{2} \times \frac{\pi}{2})$$

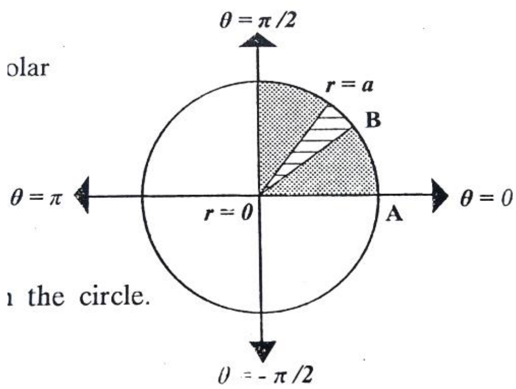
$$= \frac{\pi(b^4 - a^4)}{16}$$

Example: 4.67

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$ by transforming into polar co-ordinates.

[AU May 2011,June 2008,Nov 2007]

Solution:



Given order $dydx$ is in correct form.

Given limits are $y : 0 \to \sqrt{a^2 - x^2}$, $x : 0 \to a$

Equations are $y = 0$, $y = \sqrt{a^2 - x^2}$, $x = 0$, $x = a$

$$y^2 = a^2 - x^2$$

$x^2 + y^2 = a^2$ is a circle with centre (0,0) and radius ‘a’.

Replacement:

Put $a^2 - x^2 - y^2 = a^2 - (x^2 + y^2) = a^2 - r^2$, $dydx = rdrd\theta$

$\therefore \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$

Limits: $r : 0 \to a$, $\theta: 0 \to \frac{\pi}{2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx = \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{a^2 - r^2}\, rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^a \sqrt{a^2 - r^2}\, rdr\right) d\theta$$

Substitution:

Put $a^2 - r^2 = t$ if $r = 0 \Rightarrow t = a^2$

$-2rdr = dt$ if $r = a \Rightarrow t = 0$

$rdr = -\frac{dt}{2}$

$\therefore t : a^2 \to 0$

$$\int_0^{\frac{\pi}{2}} \left(\int_0^a \sqrt{a^2 - r^2}\, rdr\right) d\theta = \int_0^{\frac{\pi}{2}} \left[\int_{a^2}^0 \sqrt{t}\left(\frac{-dt}{2}\right)\right] d\theta$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left[\int_{a^2}^0 \sqrt{t}\, dt\right] d\theta$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left[\int_{a^2}^0 t^{\frac{1}{2}} dt\right] d\theta$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_{a^2}^0 d\theta$$

$$= -\frac{1}{2} \times \frac{2}{3} \int_0^{\frac{\pi}{2}} \left[t^{\frac{3}{2}}\right]_{a^2}^0 d\theta$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} \left(0 - (a^2)^{\frac{3}{2}}\right) d\theta$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} - (a^3)\, d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{a^3}{3} (\theta)_0^{\frac{\pi}{2}}$$

$$= \frac{a^3}{3} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi a^3}{6}$$

Exercise: 4.9

Evaluate the following by changing into polar co-ordinates.

1. $\int_0^{2a} \int_0^{\sqrt{2x-x^2}} dydx$ **Ans:** $\frac{\pi a^2}{2}$

2. $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2) dydx$ **Ans:** $\frac{\pi a^4}{8}$

3. $\int_0^1 \int_{x^2}^{2-x} xy\, dxdy$ **Ans:** $\frac{3}{8}$

4. $\int_0^a \int_y^a \frac{x}{x^2+y^2} dxdy$ **Ans:** $\frac{\pi a}{4}$

5. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \frac{x}{x^2+y^2} dxdy$ **Ans:** $\frac{\pi a}{2}$

6. $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2) dydx$ **Ans:** $\frac{\pi a^4}{4}$

7. $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y + y^3) dxdy$ **Ans:** $\frac{a^5}{5}$

8. $\int_0^1 \int_0^{\sqrt{2x-x^2}} (x^2 + y^2) dydx$ **Ans:** $\frac{3\pi}{8} - 1$

9. $\iint \frac{x^2y^2}{x^2+y^2}$ dxdy over the annular region between the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 4$

Ans: 15π

10. $\iint \frac{xy}{x^2+y^2}$ dxdy over the positive quadrant of the circle $x^2 + y^2 = a^2$ **Ans:** $\frac{a^3}{6}$

Change of Variables in Triple Integral

4.9(b) Change of variables from Cartesian co- ordinates to cylindrical co – ordinates.

To convert from Cartesian to cylindrical polar coordinates system we have the following transformation.

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

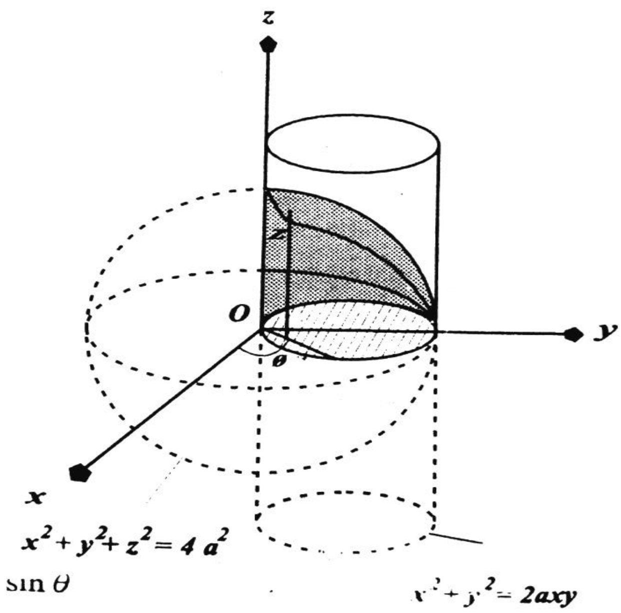
Hence the integral becomes

$$\iiint f(x,y,z)\, dzdydx = \iiint f(r,\theta,z)\, dzdrd\theta$$

Example: 4.68

Find the volume of a solid bounded by the spherical surface $\mathbf{x^2 + y^2 + z^2 = 4a^2}$ and the cylinder $\mathbf{x^2 + y^2 - 2ay = 0}$.

Solution:



Cylindrical co – ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

The equation of the sphere $x^2 + y^2 + z^2 = 4a^2$

$$r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 4a^2$$

$$r^2 + z^2 = 4a^2$$

And the cylinder $x^2 + y^2 - 2ay = 0$

$$x^2 + y^2 = 2ay$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2a\, r\sin\theta$$

$$r^2 = 2ar\sin\theta$$

$$r = 2a\sin\theta$$

Hence, the required volume,

$$\text{Volume} = \int\int\int dx\, dy\, dz$$

$$= \int\int\int r\, d\theta\, dr\, dz$$

$$= 4\int_0^{\pi/2}\int_0^{2a\sin\theta}\int_0^{\sqrt{4a^2-r^2}} r\, dz\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{2a\sin\theta} r\sqrt{4a^2-r^2}\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left[-\frac{1}{3}(4a^2-r^2)^{3/2}\right]_0^{2a\sin\theta} d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\left[-(4a^2 - 4a^2\sin^2\theta)^{3/2} + 8a^3\right] d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}(-8a^3\cos^3\theta + 8a^3)\, d\theta$$

$$= \frac{4}{3}\ 8a^3 \int_0^{\pi/2} (1 - \cos^3\theta)\, d\theta$$

$$= \frac{32\, a^3}{3} \left[\frac{\pi}{2} - \frac{2}{3}\right] \text{ cubic units}$$

Example: 4.69

Find the volume of the portion of the cylinder $x^2 + y^2 = 1$ intercepted between the plane $x = 0$ and the paraboloid $x^2 + y^2 = 4 - z$.

Solution:

Cylindrical co – ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Given $x^2 + y^2 = 1$

$$r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$r^2 = 1$$

$$r = \pm 1$$

Given $x^2 + y^2 = 4 - z$

$$r^2\cos^2\theta + r^2\sin^2\theta = 4 - z$$

$$r^2 = 4 - z$$

$$z = 4 - r^2$$

Hence the required volume

$$\text{Volume} = \int\int\int r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\, [z]_0^{4-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\,(4 - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (4r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^4}{4}\right] d\theta$$

$$= \int_0^{2\pi} \left[\left(2 - \frac{1}{4}\right) - (0 - 0)\right] d\theta$$

$$= \int_0^{2\pi} \frac{7}{4}\, d\theta$$

$$= \frac{7}{4}\, [\theta]_0^{2\pi}$$

$$= \frac{7}{4}\, [2\pi - 0] \quad = \frac{7}{2}\,\pi \text{ cubic.units}$$

Example: 4.70

Find the volume bounded by the paraboloid $\mathbf{x^2 + y^2 = az}$, and the cylinder $\mathbf{x^2 + y^2 = 2ay}$ and the plane $\mathbf{z = 0}$

Solution:

Cylindrical co – ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

The equation of the sphere $x^2 + y^2 += az$

$$r^2\cos^2\theta + r^2\sin^2\theta = az$$

$$r^2 = az$$

And the cylinder $x^2 + y^2 = 2ay$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2a\,r\sin\theta$$

$$r^2 = 2ar\sin\theta$$

$$r = 2a\sin\theta$$

Hence, the required volume,

$$\text{Volume} = \int\int\int dx\,dy\,dz$$

$$= \int\int\int r\,d\theta\,dr\,dz$$

$$= \int_0^{\pi}\int_0^{2a\sin\theta}\int_0^{\frac{r^2}{a}} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{2a\sin\theta} [z]_0^{\frac{r^2}{a}} r dr d\theta$$

$$= \int_0^{\pi}\int_0^{2a\sin\theta} \left[\frac{r^3}{a}\right] dr d\theta$$

$$= \frac{1}{a}\int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\pi}\frac{16a^4\sin^4\theta}{4}d\theta$$

$$= 4a^3 \times 2\int_0^{\pi/2}\sin^4\theta d\theta$$

$$= 4a^3 \times 2\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \quad = \frac{3\pi a^3}{2}$$

4.9(c)Change of variables from Cartesian Co – ordinates to spherical Polar Co – ordinates

To convert from Cartesian to spherical polar co-ordinates system we have the following transformation

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2 \sin\theta$$

Hence the integral becomes

$$\iiint f(x, y, z)\, dzdydx = \iiint f(r, \theta, z) r^2 \sin\theta\, drd\theta d\varphi$$

Example: 4.71

Evaluate $\int \int \int \frac{1}{\sqrt{1-x^2-y^2-z^2}}$ **dx dy dz over the region bounded by the sphere** $\mathbf{x^2 + y^2 + z^2 = 1.}$

Solution:

Let us transform this integral in spherical polar co – ordinates by taking

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r\cos\theta$$

$$dx\, dy\, dz = (r^2 \sin\theta)\, dr\, d\theta\, d\phi$$

Hence ϕ varies from 0 to 2π

ϕ varies from 0 to π

ϕ varies from 0 to 1

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \frac{1}{\sqrt{1-r^2}} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \left[\int_0^{2\pi} d\phi\right] \left[\int_0^{\pi} \sin\theta\, d\theta\right] \left[\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr\right]$$

$$= [\phi]_0^{2\pi} [-\cos\theta]_0^{\pi} \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$$

$$= (2\pi - 0)\,(1+1) \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$$

$$= 4\pi \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$$

Put $r = \sin t$; $dr = \cos t\, dt$

$r = 0 \Rightarrow t = 0$

$r = 1 \Rightarrow t = \frac{\pi}{2}$

$$= 4\pi \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}} \cos t\, dt$$

$$= 4\pi \int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{\cos^2 t}} \cos t\, dt$$

$$= 4\pi \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\, dt$$

$$= 4\pi \int_0^{\pi/2} \sin^2 t\, dt$$

$$= 4\pi \frac{1}{2}\frac{\pi}{2} = \pi^2$$

Example: 4.72

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$

Solution:

Given □ varies from 0 to 1

y varies from 0 to $\sqrt{1-x^2}$

z varies from $\sqrt{x^2+y^2}$ to 1

Let us transform this integral into spherical polar co – ordinates by using

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$dx\,dy\,dz = (r^2\sin\theta)\,dr\,d\theta\,d\phi$$

Let $z = \sqrt{x^2+y^2}$

$\Rightarrow z^2 = x^2 + y^2$

$\Rightarrow r^2\cos^2\theta = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi$

$\Rightarrow \cos^2\theta = \sin^2\theta$ $\quad [\because \cos^2\phi + \sin^2\phi = 1]$

$\Rightarrow \theta = \frac{\pi}{4}$

Let $z = 1$

$\Rightarrow r\cos\theta = 1$

$\Rightarrow r = \frac{1}{\cos\theta}$

$\Rightarrow r = \sec\theta$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant.

Limits of r : $r = 0$ to $r = \sec\theta$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{4}$

Limits of ϕ : $\phi = 0$ to $\phi = \frac{\pi}{2}$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} \frac{1}{r} r^2\sin\theta\,dr\,d\theta\,d\phi \qquad = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} r\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\left[\sin\theta\,\frac{r^2}{2}\right]_0^{\sec\theta} d\theta\,d\phi \qquad = \int_0^{\pi/2}\int_0^{\pi/4}\left[\frac{\sec^2\theta\sin\theta - 0}{2}\right] d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\frac{1}{2}\sec\theta\tan\theta\,d\theta\,d\phi \qquad = \left[\frac{1}{2}\int_0^{\pi/2} d\phi\right]\left[\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta\right]$$

$= \frac{1}{2} [\, \theta \,]_0^{\pi/2} \, [\sec\theta]_0^{\pi/4}$

$= \frac{1}{2} \left[\frac{\pi}{2} - 0\right] \left[\sqrt{2} - 1\right]$

$= \frac{\pi}{4} \left(\sqrt{2} - 1\right)$

Example: 4.73

Evaluate $\int\int\int (x^2 + y^2 + z^2)\,dx\,dy\,dz$ taken over the region bounded by the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Let us convert the given integral into spherical polar co – ordinates.

$$x = r\sin\theta\cos\phi \quad \Rightarrow x^2 = r^2\sin^2\theta\cos^2\phi$$

$$y = r\sin\theta\sin\phi \quad \Rightarrow y^2 = r^2\sin^2\theta\sin^2\phi$$

$$z = r\cos\theta \quad \Rightarrow z^2 = r^2\cos^2\theta$$

$$dx\,dy\,dz = (r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$\int\int\int (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2\,(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

Limits of r : $r = 0$ to $r = 1$

Limits of θ : $\theta = 0$ to $\theta = \pi$

Limits of ϕ : $\phi = 0$ to $\phi = 2\pi$

$$\int\int\int (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2\,(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

$$= \left[\int_0^1 r^4\,dr\right]\left[\int_0^{\pi}\sin\theta\,d\theta\right]\left[\int_0^{2\pi} d\phi\right]$$

$$= \left[\frac{r^5}{5}\right]_0^1 [-\cos\theta]_0^{\pi}\,[\phi]_0^{2\pi}$$

$$= \left(\frac{1}{5} - 0\right)(1 + 1)\,(2\pi - 0)$$

$$= \left(\frac{1}{5}\right)(2)\,(2\pi) = \frac{4\pi}{5}$$